

॥ यावदूनं तावदूनीकृत्य वर्गं च योजयेत् ॥

Vedic Mathematics

Jagadguru Swāmī Śrī
Bhāratī Kṛṣṇa Tīrthajī Mahārāja



The Author
Jagadguru Śaṅkarācārya
Śrī Bharati Kṛṣṇa Tirtha Mahārāja
(1884—1960)

VEDIC MATHEMATICS

OR

Sixteen simple Mathematical Formulae from *the Vedas*

(For One-line Answers to all Mathematical Problems)

By

JAGADGURU
SWĀMĪ ŚRĪ BHĀRATĪ KṚṢṆA TĪRTHAJĪ MAHARAJA,
ŚAṄKARĀCĀRYA OF GOVARDHANA MAṬHA, PURI

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PUBLICATION ANNOUNCEMENT

I have great pleasure in associating myself with the publication of the book Vedic Mathematics or 'Sixteen Simple Mathematical Formulae,' by Jagadguru Swami Bharati Krishna Tirtha, Shankaracharya of Govardhana Pitha. It was long awaited by his disciples. Shrimati Manjula Devi, sole inheriter of Swamiji's right, entered into an agreement with the Banaras Hindu University to publieh it and the same is now being done in the Nepal Endowment Hindu Vishvavidyalaya Sanskrit Granthamala.

I feel grateful to all those who have worked for it. Shri Arvind N. Mafatlal business magnate of Bombay and a devotee of Swamiji has taken interest in the publication of the work. He has taken the trouble of being personally present in this ceremony of publication (*Prakashana Utsava*). He has given expression to his deep devotion to Shri Shankaracharyaji by consenting to found a chair at the Banaras Hindu University by the name of Shri Jagadguru Bharati Krishna Tirtha Shankaracharya Chair of Vedic Studies for which he is making a magnificent endowment. As Vice-Chancellor of this University I accept the donation and offer my heart-felt thanks to him for his generosity.

N. H. BHAGWATI

Date* 27-3-65** ***Vim-Chancellor

Banaras Hindu University

GENERAL EDITOR'S FOREWORD

The work entitled VEDIC MATHEMATICS or 'Sixteerr Simple Mathematical Formulæ from the Vedas' was written by His Holiness Jagadguru Śaṅkarācārya Śrī Bhāratī Kṛṣṇa Tīrthajī Mahārāja of Govardhana Maṭha, Puri (1884-1960). It forms a class by itself not pragmatically conceived and worked out as in the case of other scientific works, but the result of the intuitional visualisation of fundamental mathematical truths and principles during the course of eight years of highly concentrated mental endeavour on the part of the author and therefore appropriately given the title of "mental" mathematics appearing more as miracle than the usual approach of hard-baked science, as the author has himself stated in the Preface.

Swāmī Śaṅkarācārya was a gifted scholar on many fronts of learning including science and humanities but his whole milieu was something of a much higher texture vis, that he was a Ṛṣi fulfilling the ideals and attainments of those Seers of ancient India who discovered the cosmic laws embodied in tlie Vedas. Swāmī Bhāratī Kṛṣṇa Tirtha had the same reverential approach towards the Vedas. The question naturally arises as to whether the Sūtras which form the basis of this treatise exist anywhere in the Vedic literature as known to us. But this criticism loses all its force if we inform ourselves of the definition of Veda given by Sri Śaṅkarācārya himself as quoted below:

"The very word 'Veda' has this derivational meaning i.e. the fountain-head and illimitable store-house of all knowledge. This derivation, in effect, means, connotes and implies that the Vedas *should contain* (italics mine) within themselves all the knowledge needed by mankind relating pot only to the so-called 'spiritual' (or other-worldly) matters but also to those usually described as purely 'secular', 'temporal', or 'worldly' and also to the means required by humanity as such for the achievement of all-round, complete and perfect success in all conoeivable directions and that there can be no adjectival or restrictive epithet calculated (or tending) to limit that knowledge down in any sphere, any direction or any respect whatsoever.

"In other words, it connotes and implies that our ancient Indian Vedic lore *should be* (italics mine) all-round, complete and perfect and able to throw the fullest necessary light on all matters which any aspiring seeker after knowledge can possibly seek to he enlightened on".

It is the whole essence of his assessment of Vedic tradition that it is not to be approached from a factual standpoint but from the ideal standpoint viz, as the Vedas as traditionally accepted in India as the repository of all knowledge *should be* and not what they are in human possession. That approach entirely turns the tables on all critics, for the authorship of Vedic mathematics then need not be laboriously searched in the texts as preserved from antiquity. The Vedas are well known as four in number Rk, Yaju, Sāma and Atharva but they have also the four Upavedas and the six Vedāṅgas all of which form an indivisible corpus of divine knowledge as it once was and as it may be revealed. The four Upavedas are as follows:—

Veda	*Upaveda*
Rgveda	Āyurveda
Sāmaveda	Gāndharvaveda
Yajurveda	Dhaṅurveda
Atharvaveda	Sthāpathyaveda

In this list the Upaveda of Sthāpatya or engineering comprises all kinds of architectural and structural human endeavour and all visual arts. Swamiji naturally regarded mathematics or the science of calculations and computations to fall under this category.

In the light of the above definition and approach must be understood the author's statement that the sixteen Sūtras on which the present volume is based form part of a Pariśiṣṭa of the Atharvaveda. We are aware that each Veda has its subsidiary apocryphal texts some of which remain in manuscripts and others have been printed but that formulation has not closed. For example, some Pariśiṣṭas of the Atharvaveda were edited by G. M. Rolling and J. Von Negelein, Liepzing, 1909-10. But this work of Śrī Śaṅkarāçāryaji deserves to be regarded as a new Pariśiṣṭa by itself and it is not surprising that the Sūtras

mentioned herein do not appear in the hitherto known Pariśiṣṭas.

A list of these main 16 Sūtras and of their sub-sūtras or corollaries is prefixed in the beginning of the text and the style of language also points to their discovery by Śrī Swāmījī himself. At any rate, it is needless to dwell longer on this point of origin since the vast merit of these rules should be a matter of discovery for each intelligent reader. Whatever is written here by the author stands on its own merits and is presented as such to the mathematical world.

Swamiji was a marvellous person with surpassing qualities and was a prolific writer and eloquent speaker. I had the good fortune of listening to his discourses for weeks together on several occasions when he used to visit Lucknow and attracted large audiences. He could at a stretch speak for several hours in Sanskrit and English with the same facility and the intonation of his musical voice left a lasting impression on the minds of hie hearers. He was an ardent admirer of Bhartrhari the great scientific thinker of the Golden Age of Indian history in a different field viz, that of philosophy of grammar.

Swamiji had planned to write 16 volumes on all aspects and branches of mathematical processes and problems and there is no doubt that his mental powers were certainly of that calibre, but what has been left to us is this introductory volume which in itself is of the highest merit for reason of presenting a new technique which the author styles as "mental" mathematics different from the orthodox methods of mathematicians all over the world. Arithmetical problems usually solved by 18, 28 or 42 steps in case of such vulgar fractions as 1/19, 1/29, 1/49 are here solved in one simple line and that is possible to be done even by young boys. The truth of these methods was demonstrated by this saintly teacher before many University audiences in India and in the U.S.A. including learned Professors and every one present was struck with their originality and simplicity.

We are told in his Preface by Swāmī Śaṅkarācārya that he contemplated to cover all the different branches of mathe-

matics such as arithmetic, algebra, geometry (plane and solid) trigonometry (plane and spherical) conics—geometrical and analytical, astronomy, calculus—differential and integral etc., with these basic Sūtras. That comprehensive application of the Sūtras could not be left by him in writing but if some one has the patience and the genius to pursue the method and implications of these formulae he may probably be able to bring these various branches within the orbit of this original style.

A full fledged course of his lecture-demonstrations was organised by the Nagpur University in 1952 and some lectures were delivered by Swamiji at the B.H.U. in 1949. It is, therefore, in the fitness of things and a happy event for the B.H.U. to be given the opportunity of publishing this book by the courtesy of Srimati Manjula Devi Trivedi, disciple of Srī Swāmījī who agreed to make over this manuscript to us through the efforts of Dr. Pt. Omkarhath Thakur. The work has been seen through the Press mainly by Dr. Prem Lata Sharma, Dean, Faculty of Music & Pine Arts in the University. To all of these our grateful thanks are due. Dr. Brij Mohan, Head of the Department of Mathematics, B.H.U., took the trouble, at my request, of going through the manuscript and verifying the calculations for which I offer him my best thanks. I also express gratitude to Sri Lakshmidas, Manager, B.H.U. Press, for taking great pains in printing this difficult text.

We wish to express our deepest gratitude to Śrī Swāmī Pratyagātmānanda Saraswati for the valuable foreword that he has written for this work. Today he stands pre-eminent in the world of Tantric scholars and is a profound mathematician and scientific thinker himself. His inspiring words are like fragrant flowers offered at the feet of the ancient Vedic Ṛśis whose spiritual lineage was revealed in the late Śankarācārya Śrī Bhāratī Kṛṣṇa Tirtha. Swāmī Pratyagātmānandajī has not only paid a tribute to Śrī Śankarācāryajī but his ambrocial words have showered blessings on all those who are lovers of intuitional experiences in the domain of metaphysics and physics. Swamiji, by a fortunate chance, travelled from Calcutta

to Varanasi to preside over the Tantric Sammelan of the Varanaseya Sanskrit University (8th to 11th March 1965) and although he is now 85 years of age, his innate generosity made him accept our request to give his foreword.

I am particularly happy that I am able to publish this work under the Nepal Endowment Hindu Vishvavidyalaya Publication Series, for I cntcrtained an ardent desire to do so since our late President Dr. Rajcndra Prasadji spoke to me about its existence when I once met him in New Delhi in the lifetime of Śrī Swiimiji.

V. S. ACRAWALA,
M.A., **Ph.D., D.Litt.**
General Editor,
Hindu Vishwavidyalaya
Nepal Rajya Sanskrit
Granthamala Series.

Banaras Hindu University
Varanasi-5
March 17, 1965.

FOREWORD

Vedic Mathematics by the late Śaṅkarācārya (Bhāratī Kṛṣṇa Tīrtha) of Govardhana Pīṭha is a monumental work. In his deep-layer explorations of cryptic Vedic mysteries relating specially to their calculus of shorthand formulae and their neat and ready application to practical problems, the late Śaṅkarācārya shews the rare combination of the probing insight and revealing intuition of a Yogi with the analytic acumen and synthetic talent of a mathematician. With the late Śaṅkarācārya we belong to a race, now fast becoming extinct, of die-hard believers who think that the Vedas represent an inexhaustible mine of profoundest, wisdom in matters both spiritual and temporal ; and that.this store of wisdom was not, as regards its assets of fundamental validity and value at least, gathered by the laborious inductive and deductive methods of ordinary systematic enquiry, but was a direct gift of revelation to seers and sages who in their higher reaches of Yogic realization were competent to receive it from a Source, perfect and immaculate. But we admit, and the late Śaṅkarācārya has also practically admitted, that one cannot expect to convert or revert criticism, much less carry conviction, by merely asserting one's staunchest beliefs. To meet these ends, one must be prepared to go the whole length of testing and verification by accepted, accredited methods. The late Śaṅkarācārya has, by his comparative and critical study of Vedic mathematics, made this essential requirement in Vedic studies abundantly clear. So let us agree to gauge Vedic mysteries not as we gauge the far-off nabulae with the poet's eye or with that of the seer, but with the alert, expert, scrutinizing eye of the physical astronomer, if we may put it as that.

That there is a consolidated metaphysical background in the Vedas of the objective sciences including mathematics as regards their basic conceptions is a point that may be granted by a thinker who has looked broadly and deeply into both the realms.

In our paper recently published—'The Metaphysics of Physics'—we attempted to look into the mysteries of creative emergence as contained in the well-known cosmogenic Hymn

(Rg. X.190) with a view to unveiling the metaphysical background where both ancient wisdom and modern physics may meet on a common basis of logical understanding, and compare notes, discovering, where possible, points of significant or suggestive parallelism between the two sets of concepts, ancient and modern. That metaphysical background includes mathematics also; because physics as ever pursued is the application of mathematics to given or specified space-time-event situations. There we examined Tapas as a fundamental creative formula whereby the Absolute emerges into the realms of measures, variations, limits, frame-works and relations. And this descent follows a logical order which seems to lend itself, within a framework of conditions and specifications, to mathematical analysis. *Rātri* in the Hymn represents the Principle of Limits, for example, *Ṛtañca Satyañca* stand for Becoming (*Calana-kalana*) and Being (*vartana-kalana*) at a stage where limits or conditions or conventions do not yet arise or apply. The former gives the unconditioned, unrestricted how or *thus* of cosmic process; the latter, what or *that* of existence. ***Tapas,*** which corresponds to *Ardhamātrā* in Tantric symbolism, negotiates, in its role specially of critical variation, between what is, *ab-initio,* unconditioned and unrestricted, and what appears otherwise, as for instance, in our own universe of logico-mathematical appreciation.

This is, necessarily, abstruse metaphysics, but it is, nevertheless, the starting background of both physics and mathematics. But for all practical purposes we must come down from mystic nabulae to the *terra firma* of our actual apprehension and appreciation. That is to say, we must descend to our own pragmatic levels of time-space-event situations. Here we face actual problems, and one must meet and deal with these squarely without evasion or mystification. The late Śaṅkarācārya has done this masterly feat with an adroitness that compels admiration.

It follows from the fundamental premises that the universe we live in must have a basic mathematical structure, and consequently, to know a fact or obtain a result herein, to any required degree of precision, one must obey the rule of mathe-

matical measures and relations. This, however, ore may do consciously or semi-consciously, systematically or haphazardly. Even some species of lower animals are by instinct gifted mathematicians ; for example, the migratory bird which flies thousands of miles off from its nest-home, and after a period, unerringly returns. This 'implies a subconscious mathematical talent that works wonder. We may cite the case of a horse who was a mathematical prodigy and could 'tell' the result of a cube root (requiring 32 operations. according to M. Materlink in his 'Unknown Quest') in a twinkle of the eye. This sounds like magic, but it is undeniable that the feat of mathematics does sometimes assume a magical look. Man, undoubtedly, has been given his share of this magical gift. And he can improve upon it by practice and discipline, by Yoga and allied methods. This is undeniable also. Lately, he has devised the 'automatic brain' for complicated calculations by science, that looks like magic.

But apart from this 'magic', there is and has been, the 'logic' of mathematics also. Man works from instinct, talent, or even genius. But ordinarily he works as a logical entity requiring specified data or premises to start from, and more or less elaborate steps of reasoning to arrive at a conclusion. This is his normal process of induction and deduction. Here formulæ (*Sūtras*) and relations (e.g. equations) must obtain as in mathematics. The magic and logic of mathematics in some cases get mixed up ; but it is sane to keep them apart. You can get a result by magic, but when you are called upon to prove, you must have recourse to logic.

Even in this latter case, your logic (your formulæ and applications) may be either simple and elegant or complicated and cumbrous. The former is the ideal to aim at,. We have classical instances of master mathematicians whose methods of analysis and solution have been regarded as marvels of cogency, compactness and clcgancc. Some have been 'beautiful' as a poem (e.g. Lagrange's 'Analytical Mechanics.')

The late Śaṅkarācārya has claimed, and rightly we may think, that the Vedic *Sūtras* and their applications possess these

virtues to a degree of eminence that cannot be challenged. The outstanding merit of his work lies in his actual proving of this contention.

Whether or not the Vedas be believed as repositories of perfect wisdom, it is unquestionable that the Vedic race lived *not* as merely pastoral folk possessing a half-or-quarter-developed culture and civilization. The Vedic seers were, again, not mere 'navel-gazers' or 'nose-tip-gazers'. They proved themselves adepts in all levels and branches of knowledge, theoretical and practical. For example, they had their varied objective science, both pure and applied.

Let us take a concrete illustration. Suppose in a time of drought we require rains by artificial means. The modern scientist has his own theory and art (technique) for producing the result. The old seer scientist had his both also, but different from these now availing. He had his science and technique, called *Yajña,* in which *Mantra, Yantra* and other factors must co-operate with mathematical determinateness and precision. For this purpose, he had developed the six auxiliaries of the Vedas in each of which mathematical skill and adroitness, occult or otherwise, play the decisive role. The *Sūtras* lay down the shortest and surest lines. The correct intonation of the *Mantra,* the correct configuration of the *Yantra* (in the making of the *Vedī* etc., e.g. the quadrature of a circle), the correct time or astral conjugation factor, the correct rhythms etc., all had to be perfected so as to produce the desired result effectively and adequately. Each of these required the calculus of mathematics. The modern technician has his logarithmic tables and mechanics' manuals; the old Yājñika had his *Sūtras.* How were the *Sūtras* obtained ?—by magic or logic or both ?—is a vital matter we do not discuss here. The late Śaṅkarācārya has claimed for them cogency, compactness and simplicity. This is an even more vital point, and we think, he has reasonably made it good.

Varanasi,
22-3-1965

SWAMI PRATYAGATMANANDA
SARASWATI

A HUMBLE HOMAGE

The late Śankarācārya's epoch-making work on Vedic-Mathematics brings to the notice of the intelligentsia most strikingly a new theory and method, now almost unknown, of arriving at the truth of things which in this particular case concerns the truth of numbers and magnitude, but might as well cover, as it undoubtedly did in a past age in India, all sciences and arts, with results which do not fail to evoke a sense of awe and amazement today. The method obviously is radically differnt from the one adopted by the modern mind.

Music and not Mathematics is my field (although the philosophy of numbers, cosmic and metaphysical correspondences with musical numbers, the relation of numbers with consonant, dissonant and assonant tonal intervals etc., closely inter-relate music and mathematics), but study of the traditional literature on music and fine arts with which I have been concerned for the last few years has convinced me of one fundamental fact regarding the ancient Indian theory and method of knowledge and experience *vis a vis* the modern. While all great and true knowledge is born of intuition and not of any rational process or imagination. there is a radical difference between the ancient Indian method and the modern Western method concerning intuition.

The divergence embraces everything other than the fact of intuition itself—the object and field of intuitive vision, the method of working out experience and rendering it to the intellect. The modern method is to get the intuition by suggestion from an appearance in life or nature or from a mental idea and even if the source of the intuition is the soul, the method at once relates it to a support external to the soul. The ancient Indian method of knowledge had for its business to disclose something of the Self, the Infinite or the Divine to the regard of the soul—the Self through its expressions, the infinite through its finite symbols and the Divine through his powers. The

process was one of Integral knowledge and in its subordinate ranges was instrumental in revealing the truths of cosmic phenomena and these truths were utilised for worldly ends.

These two methods are based on different theories of knowledge and experience, fundamentally divergent in outlook and approach. The world as yet knows very little of the ancient Indian method, much less of its secret techniques. Sri Śaṅkarācārya's remarkably unique work of Vedic mathematics has brought to popular notice demonstrably for the first time that the said method was usefully employed in ancient India in solving problems of secular knowledge just as for solving those of the spiritual domain.

I am happy that in the printing and publication of this monumental work and the preceding spade-work I had the privilege to render some little service.

PREM LATA SHARMA
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Varanasi-5.
23-3-65.

CONVENTIONAL TO UNCONVENTIONALLY ORIGINAL

This book ***Vedic Mathematics*** deals mainly with various vedic mathematical formulae and their applications for carrying out tedious and cumbersome arithmetical operations, and to a very large extent, executing them mentally. In this field of mental arithmetical operations the works of the famous mathematicians Trachtenberg and Lester Meyers (High Speed Maths) are elementary compared to that of **Jagadguruji.**

Some **people** may find it **difficult,** at first reading, to understand the arithmetical operations although they have been explained very lucidly by **Jagadguruji**. It is not because the explanations are lacking in any manner but because the methods are totally unconventional. Some people are so deeply rooted in the conventional methods that they, probably, subconsciously reject to see the logic in unconventional methods.

An attempt has been made in this note to explain the unconventional aspects of the methods. Once the reader gets used to the unconventional in the beginning itself, he would find no difficulty in the later chapters. Therefore the explanatory notes are given for the first few chapters only.

Chapter I

Chapter I deals with a topic that has been dealt with comprehensively in the chapter 26 **viz. 'Recurring Decimal'.** Gurudeva has discussed the recurring decimals of 1/19, 1/29, etc. in chapter I to arouse curiosity and create interest. In conversion of vulgar fractions into their decimal equivalents Gurudeva has used very unconventional methods of multiplication and division.

In calculation of decimal equivalent of 1/19, first method of the **'Ekādhika Sūtra'** requires multiplication of 1 by 2 by a special and unconventional process. Inconventional method product of 1, the multiplicand, by 2 the multiplier, is 2 and that is the end of multiplication process. It is not so in the unconventional **'Ekādhika'** method. In this method, in the above example, 1 is the first multiplicand and its product with multiplier '2' is 2 which in this special process becomes the second multiplicand. This when multiplied by the multiplier (which remains the same) 2 gives the product as 4 which becomes the third multiplicand. And the process of

multiplication thus goes on till the digits start recurring.

Similarly in the second method of the **'Ekādhika Sūtra'** for calculating the decimal equivalent of **1/19, it** is required to divide I by 2 by an unconventional and special process. In the conventional method when 1, the dividend, is to be divided by the divisor '2', the quotient is 0.5 and the process of division ends. In the special method of **'Ekādhika Sūtra'** for calculating decimal equivalents, the process starts by putting zero as the first digit of the quotient, 1 as the first remainder. **A** decimal point is put after the first quotient digit which is zero. Now, the first remainder digit '1' is prefixed to the first quotient digit '0' to form '10' as the second dividend. Division of 10 by the divisor 2 (which does not change) gives 5 as the second quotient digit which is put after the decimal point. The second remainder digit '0' is prefixed to the second quotient digit 5 to form 5 as the third dividend digit. Division of 5 by 2 gives 2 as the third quotient digit and 1 as the third remainder digit which when prefixed to the third quotient digit '2' gives 12 as the fourth dividend and so the process goes on till the digits start recurring.

Chapter III

Vinculum is an ingenious device to reduce single digits larger than 5, thereby **facilitatingmultiplication** specially for **the mental-**one-line method. Vinculum method is based on the fact that 18 is same as (20-2) and 76 as (100-24) or 576 as (600-24). **Guru-**deva has made this arithmetical fact a powerful device by writing 18 as $2\bar{2}$, 76 as $1\bar{2}4$ arid 576 as $6\bar{2}\bar{4}$. This device is specially useful in vedic division method.

A small note on **'aliquot'** may facilitate the study for some. Aliquot part is the part contained by the whole an integral number of times, **e.g.** 12 is contained by the whole number **110,** 9 times. or in simple words it is the quotient of that fraction.

Chapter IV

In the division by the **Nikhilam' method** the dividend is divided into two portions by a vertical line. This vertical line should have as many digits to its right as there can be in the highest possible remainder. In general the number of such digits are the same as in the figure which is one less than the divisor. Needless to state that the vertical and horizontal lines must be drawn neatly when using this method.

WING. COM. VISHVA MOHAN **TIWARI**

CONTENTS

MY BELOVED GURUDEVA

SMTI. MANJULA TRIVEDI

[In *the* lines that follow *the* writer gives a *short* biographical *sketch* of *the* illustrious author of *Vedic* Mathematics and a *short account* of *the* genesis of his work *now* published, based on intimate personal *knowledge*—EDITOR.]

Very few persons can there be amongst the cultured people of India who have not heard about HIS HOLINESS JAGADGURU SHANKARACHARYA SRI BHARATI KRISHNA TIRTHAJI MAHARAJ, the magnificent and divine personality that gracefully adorned the famous Govardhan Math, Puri, his vast and versatile learning, his spiritual and educational attainments, his wonderful research achievements in the field of Vedic Mathematics and his consecration of all these qualifications to the service of humanity as such.

His Holiness, better known among his disciples by the beloved name 'Jagadguruji' or 'Gurudeva' was born of highly learned and pious parents in March, 1884. His father, late Sri P. Narasimha Shastri, was then in service as a Tahsildar at Tinnivelly (Madras Presidency) who later retired as a Deputy Collector. His uncle, late Shri Chandrashekhar Shastri, was the Principal of the Maharaja's College, Vizianagaram and his great-grandfather was late Justice C. Ranganath Shastri of the Madras High Court.

Jagadguruji, named as Venkatraman in his early days, was an exceptionally brilliant student and invariably won the first place in all the subjects in all the classes throughout his educational career. During his school days, he was a student of National College, Trichanapalli; Church Missionary Society College, Tinnevelli and Hindu College, Tinnevelli. Be passed his matriculation examination from the Madras University in January, 1899, topping the list as usual.

He was extra-ordinarily proficient in Sanskrit and oratory and on account of this he was awarded the title of 'SARASWATI'

by the Madras Sanskrit Association in July, 1899 when he was still in his 16th year. One cannot fail to mention at this stage the profound impression left on him by his Sanskrit Guru Shri Vedam Venkatrai Shastri whom Jagadguruji always remembered with deepest love, reverence and gratitude, with tears in his eyes.

After winning the highest place in the B.A. Examination, Shri Venkatraman Saraswati appeared at the M.A. Examination of the American College of Sciences, Rochester, New York, from Bombay Centre in 1903 ; and in 1904 at the age of just twenty he passed M.A. Examination in further seven subjects simultaneously securing the highest honours in all, which is perhaps the all-time world-record of academic brilliance. His subjects included Sanskrit, Philosophy, English, Mathematics, History and Science.

As a etudent Venkatraman was marked for his splendid brilliance, superb retentive memory and ever-insatiablt curiosity. He would deluge his teachers with myriads of piercing questions which made them uneasy and forced them frequently to make a frank confession of ignorance on their part. In this respect, he was considered to be a terribly mischievous student.

Even from his University days Shri Venkatraman Saraswati had started contributing learned articles on religion, philosophy, sociology, history, politics, literature etc., to late W. T. Stead's "REVIEW OF REVIEWS" and he was specially interested in all the branches of modern science. In fact, study of the latest researches and discoveries in modern science continued to be Shri Jagadguruji's hobby till his very last days.

Sri Venkatrarnan started his public life unde the guidance of late Hon'ble Shri Gopal Krishna Gokhale, C.I.E. in 1905 in connection with the National Education Movement and the South African Indian issue. Although, however, on the one hand, Prof. Venkatraman Saraswati had acquired an endless fund of learning and his desire to learn ever more was still unquenchable and on the other hand the urge for selfless service

of humanity swayed his heart mightily, yet the undoubtedly deepest attraction that Venkatraman Saraswati felt was that towards the study and practice of the science of sciences—the holy ancient Indian spiritual science or Adhyātma-Vidyā. In 1908, therefore, he proceeded to the Sringeri Math in Mysore to lay himself at the feet of the renowned late Jagadguru Shankaracharya Maharaj Shri Satchidiinanda Sivābhinava Nrisimha Bhārati Swami:

But he had not stayed there long, before he had to assume the post of the first Principal of the newly started National College at Rajmahendri under a pressing and clamant call of duty from the nationalist leaders. Prof. Venkatraman Saraswati continued there for three years but in 1911 he could not resist his burning desire for spiritual knowledge, practice and attainment any more and, therefore, tearing himself off suddenly from the said college he went back to Shri Satchidānanda Siviibhinava Nrisimha Bhārati Swami at Sringeri.

The next eight years he spent in the profoundest study of the most advanced Vedanta Philosophy and practice of the Brahma-eadhnna. During these days Prof. Venkatraman used to study Vedanta at the feet of Shri Nrisimha Bhārati Swami, teach Sanskrit and Philosophy in schools there, and practise the highest and most vigorous Yoga-sādhāna in the nearby forests. Frequently, he was also invited by several institutions to deliver lectures on philosophy; for example he delivered a series of sixteen lectures on Shankarachnrya's Philosophy at Shankar Institute of Philosophy, Amalner (Khandesh) and similar lectures at several other places like Poona, Bombay etc.

After several years of the most advanced studies, the deepest meditation, and the highest spiritual attainment Prof. Venkatraman Saraswati was initiated into the holy order of SAṂNYASA at Banaras (Varanasi) by his Holiness Jagadguru Shankaracharya Sri Trivikram Tirthaji Maharaj of Shāradāpeeth on the 4th July 1919 and on this occasion he was given the new name, Swami Bharati Krishna Tirtha.

This was the starting point of an effulgent manifestation of Swamiji's real greatness. Within two years of his stay in-the holy order, he proved his unique suitability for being installed on the pontifical throne of Sharada Peetha Shankaracharya and accordingly in 1921, he was so installed with all the formal ceremonies despite all his reluctance and active resistance. Immediately, on assuming the pontificate Shri Jagadguruji started touring India from corner to corner and delivering lectures on Sanātana Dharma and by his scintillating intellectual brilliance, powerful oratory, magnetic personality, sincerity of purpose, indomitable will, purity of thought, and loftiness of character he took the entire intellectual and religious class of the nation by storm.

Jagadguru Shankaracharya Shri Madhusudan Tirtha of Govardhan Math Puri was at this stage greatly impressed by Jagadguruji and when the former was in failing health he requested· Jagadguruji to succeed him on Govardhan Math Gadi. Shri Jagadguruji continued to resist his importunate requests for a long time but at last when Jagadguru Shri Madhusudan Tirtha's health took a serious turn in 1925 he virtually forced Jagadguru Shri Bharati Krishana Tirthaji to accept the Govardhan Math's Gadi and accordingly Jagadguruji installed Shri Swarupanandji on the Sharadapeeth Gadi and himself assumed the duties of the ecclesiastical and pontifical head of Sri Govardhan Math, Puri.

In this capacity of Jagadguru Shankaracharya of Govardhan Math, Puri, he continued to disseminate the holy spiritual teachings of Sanatana Dharma in their pristine purity all over the world the rest of his life for 35 years. Months after months and years after years he spent in teaching and preaching, talking and lecturing, discussing and convincing millions of people all over the country. He took upon himself the colossal task of the renaissance of Indian culture, spreading of Sanatena Dharma, revival of the highest human and moral values and enkindling of the loftiest spiritual enlightenment throughout the world and he dedicated his whole life to this lofty and noble mission.

Prom his very early days Jagadguruji was aware of the need for the right interpretation of "Dharma" which he defined as "the sum total of all the means necessary for speedily making and permanently keeping all the people, individually as well as collectively superlatively comfortable, prosperous, happy, and joyous in all respects (including the physical, mental, intellectual, educational, economic, social, political, psychic, spritual etc. *ad infinitum*)". He was painfully aware of the "escapism" of some from their duties under the garb of spirituality and of the superficial modem educational varnish of the others, divorced from spiritual and moral standards. He, therefore, always laid great emphasis on the necessity of harmonising the 'spiritual' and the 'material' spheres of daily life. He also wanted to remove the false ideas, on the one hand, of those persons who think that Dharma can be practised by exclusively individual spiritual Sādhanā coupled with more honest bread-earning, ignoring one's responsibility for rendering selfless service to the society and on the other hand of those who think that the Sādhanā can be complete by mere service of society even without learning or practising any spirituality oneself. He wanted a happy blending of both. He stood for the omnilateral and all-round progress simultenaously of both the individual and society towards the speedy realisation of India's spiritual and cultural ideal, the lofty Yedantic ideal of 'Pūrṇatva' (perfection and harmony all-round).

With these ideas agitating his mind for several decades he went on carrying on a laborious, elaborate, patient and day-and-night research to evolve finally a splendid and perfect scheme for all-round reconstruction first of India and through it of the world. Consequently Sri Jagadguruji founded in 1953 at Nagpur an institution named Sri Vishwa Punarnirmana Sangha (World Reconstruction Association). The Administrative Board of the Sangha consisted of Jagadguruji's disciples, devotees and admirers of his idealistic and spiritual ideals for humanitarian service and included a number of high court judges, ministers, educationists, statesmen and other personage of the highest calibre

pleasure. To see him was a privilege. To speak to him was a real blessing and to be granted a specialinterview—Ah! that was the acme of happiness which people coveted most in all earnestness. The magnetic force of his wonderful personality was such that one word, one smile, or even one look was quite enough to convert even the most sceptic into his most ardent and obedient disciple. He belonged to all irrespective of caste or creed and he was a real Guru to the whole world.

People of all nationalities, religions and climes, Brahmins and non-Brahmins, Hindus and Mahomedans, Parsis and Christians, Europeans and Americans received equal treatment at the hands of Mis Holiness. That was the secret of the immense popularity of this great Mahatma.

He was grand in his simplicity. People would give anything and everything to get his blessings and he would talk words of wisdom as freely without fear or favour. He was most easily accessible to all. Thousands of people visited him and prayed for the relief of their miseries. He had a kind word to say to each, after attentively listening to his or her tale of woe and then give them some 'prasad' which would cure their malady whether physical or mental. He would actually shed tears when he found people suffering and would pray to God to relieve their suffering.

He was mighty in his learning and voracious in his reading. A sharp intellect, a retentive memory and a keen zest went to mark him as the most distinguished scholar of his day. His leisure moments he would never spend in vain. He was always reading something or repeating something. There was no branch of knowledge which he did not know and that also 'shastrically'. He was equally learned in Chandahsastra, Ayurveda and Jyotish Sastra. He was a poet of uncommon merit and wrote a number of poems in Sanskrit in the praise of his guru, gods and godesses with a charming flow of Bhakti so conspicuous in all his writings.

I have got a collection of over three thousand slokas forming part of the various eulogistic poems composed by Gurudeva

in adoration of various Devas and Devis. These Slokas have been edited and are being translated into Hindi. They are proposed to be published in three volumes along with Hindi translation.

The book on "Sanatana Dharma" by H. H. Swami Bhāratī Krisna Tirtha Mahārāja has been published by Bharatiya Vidya Bhavan, Bombay.

Above all, his Bhakti towards his Vidyāguru was something beyond description. He would talk for day-together about the greatness of his Vidyāguru. He would be never tired of worshipping the Guru. His Guru also was equally attached to him and called our Swamiji as the own son of the Goddess of Learning, Shri Sarada. Everyday he would first worship his guru's sandals. His "Gurupādukā Stotra' clearly indicates the qualities he attributed to the sandals of his guru.

Shri Bhāratī Kṛiṣṇa Tirtha was a great Yogin and a "Siddha" of a very high order. Nothing was impossible for him. Above all he was a true Samnyasin. He held the world but as a stage where every one had to play a part. In short, he was undoubtedly a very great Mahātmā but without any display of mysteries or occultisms.

I have not been able to express here even one millionth part of what I feel. His spotless holiness, his deep piety, his endless wisdom, his childlike peacefulness, sportiveness and innocence and his universal affection beggar all description. His Holiness has left us a noble example of simplest living and highest thinking. May all the world benefit by the example of a life so nobly and so simply, so spiritually and so lovingly lived.

Introductory Remarks on the Present Volume

I now proceed to give a short account of the genesis of the work published herewith. Revered Guruji used to say that he had reconstructed the sixteen mathematical formulae (given in this text) from the Atharvaveds after assiduous research and 'Tapas' for about eight years in the

forests surrounding Sringeri. Obviously these formulae are not to be found in the present recensions of Atharvaveda ; they were actually reconstructed, on the basis of intuitive revelation, from materials scattered here and there in the Atharvaveda. Revered Gurudeva used to say that he had written sixteen volumes (one for each Siitra) on these Sūtras and that the manuscripts of the said volumes were deposited at the house of one of his disciples. Unfortunately, the said manuscripts were lost irretrievably from the place of their deposit and this colossal loss was finally confirmed in 1956. Revered Gurudeva was not much perturbed over this irretrievable loss and used to say that everything was there in his memory and that he could re-write the 16 volums!

My late husband Sri C. M. Trivedi, Hon. Gen. Secertary V. P. Sangh noticed that while Sri Jagadguru Maharaj was busy demonstratirig before learned people and societies Vedic Mathematics as discovered and propounded by him, some persons who had grasped a smattering of the new Siitras had already started to dazzle audiences as prodigies claiming occult powers without aknowledging indebtedness to the Sūtras of Jagadguruji. My husband, therefore, pleaded earnestly with Gurudeva and persuaded him to arrange for the publication of the Siitras in his own name.

In 1957, when he had decided finally to undertake a tour of the U.S.A. he re-wrote from memory the present volume, giving an introductory account of the sixteen formulae reconstructed by him. This volume was written in his old age within one month and a half with his failing health and weak eyesight. He had planned to write subsequent volumes, but his failing health (and cataract developed in both eyes) did not allow the fulfilment of his plans. Now the present volume is the only work on Mathematics that has been left over by Revered Guruji ; all his other writings on Vedic Mathematics have, alas, been lost for ever.

The typescript of the present volume was left over by Revered Gurudeva in U.S.A. in 1958 for publication. He

had been given to understand that he would have to go to the U.S.A. for correction of proofs and personal supervision of printing. But his health deteriorated after his return to India and finally the typescript was brought back from the U.S.A. after his attainment of Mahasamadhi, in 1960.

ACKNOWLEDGEMENTS

I owe a deep debt of gratitude to Justice N. H. Blagwati, the enlightened Vice-Chancellor of the Banaras Hindu University and other authorities of the B.H.U. who have readily undertaken the publication of this work which was introduced to them by Dr. Pt. Omkarnath Thakur. I am indebted to Dr. Thakur for this introduction. My hearty and reverent thanks are due to Dr. V. S. Agrawala (Professor, Art & Architecture, B.H.U.) the vateran scholar, who took the initiative and throughout kept up a very keen interest in this publication. It is my pleasant duty to offer my heartfelt gratitude to Dr. Prem Lata Sharma, Dean, Faculty of Music and Fine Arts, B.H.U. who voluntarily took over the work of press-dressing of the typescript and proof-reading of this volume after a deadlock had come to prevail in the process of printing just at the outset. But for her hard labour which she has undertaken out of a sheer sense of reverence for the noble and glorious work of Revered Gurudeva this volume would not have seen the light of the day for a long time. I trust that Revered Gurudeva's Holy Spirit will shower His choicest blessings on her. My sincere thanks are also due to Sri S. Nijabodha of the Research Section under the charge of Dr. Sharma, who has ably assisted her in this onerous task.

The Humblest of His Disciples
Smti. MANJULA TRIVEDI
Hony. General Secretary
SRI Vishwa Punarnirmana
Sangha, Nagpur.

Nagpur,
16th March, 1965.

AUTHOR'S PREFACE

A—A DESCRIPTIVE PREFATORY MOTE ON THE ASTOUNDING WONDERS OF ANCIENT INDIAN VEDIC MATHEMATICS

1. In the course of aur discourses on manifold and multifarious subjects (spiritual, metaphysical, philosophical, psychic, psychological, ethical, educational, scientific, mathematical, historical, political, economic, social etc., etc., from time to time and from place to place during the last five decades and more, we have been repeatedly pointing out that the Vedas (the most ancient Indian scriptures, nay, the oldest "Religious" scriptures of the whole world) claim to deal with all branches of learning (spiritual and temporal) and to give the earnest seeker after knowledge all the requisite instructions and guidance in full detail and on scientifically—nay, mathematically—accurate lines in them all and so on.

2. The very word "Veda" has this derivational meaning i.e. the fountain-head and illimitable store-house of all knowledge. This derivation, in effect, means, connotes and implies that the Vedas should contain within themselves all the knowledge needed by mankind relating not only to the so-called 'spiritual' (or other-worldly) matters but also to those usually described as purely "secular", "temporal", or "wordly"; and also to the means required by humanity as such for the achievement of all-round, complete and perfect success in all conceivable directions and that there can be no adjectival or restrictive epithet calculated (or tending) to limit that knowledge down in any sphere, any direction or any respect whatsoever.

3. In other words, it connotes and implies that our ancient Indian Vedic lore should be all-round complete and perfect and able to throw the fullest necessary light on all matters which any aspiring seeker after knowledge can possibly seek to be enlightened on.

4. It is thus in the fitness of things that the Vedas include (i) *Āyurveda* (anatomy, physiology, hygiene, sanitary science, medical science, surgery etc., etc.,) not for the purpose of achieving perfect health and strength in the after-death future but in order to attain them *here and* now in our present physical bodies; (ii) *Dhanurveda* (archery and other military sciences) not for fighting with one another after our transportation to heaven but in order to quell and subdue all invaders from abroad and all insurgents from within; (iii) *Gāndharva Veda* (the science and art of music) and (iv) *Sthāpatya Veda* (engineering, architecture etc., and all branches of mathematics in general). All these subjects, be it noted, are inherent parts of the Vedas i.e. arc reckoned as "spiritual" studies and catered for as such therein.

5. Similar is the case with regard to the *Vedāṅgas* (i.e. grammar, prosody, astronomy, lexicography etc., etc.,) which, according to the Indian cultural conceptions, are also inherent parts and subjects of *Vedic (i.e. Religious) study.*

6. As a direct and unshirkable consequence of this analytical and grammatical study of the real connotation and full implications of the word "Veda" and owing to various other historical causes of a personal character (into details of which we need not now enter), we have been from our very early childhood, most earnestly and actively striving to study the Vedas critically from this stand-point and to realise and prove to ourselves (and to others) the correctness (or otherwise) of the derivative meaning in question.

7. There were, too, certain personal historical reasons why in our quest for the discovering of all learning in all its departments, branches, sub-branches etc., in the Vedas, our gaze was riveted mainly on ethics, psychology and metaphysics on the one hand and on the "positive" sciences and especially mathematics on the other.

8. And the contemptuous or, at best patronising attitude adopted by some so-called Orientalists, Indologists, antiquarians, research-scholars etc., who condemned, or light-

heartedly, nay; irresponsibly, frivolously and flippantly dismissed, several abstruse-looking and recondite parts of the Vedas as "sheer-nonsense"—or as "infant-humanity's prattle", and so on, merely added fuel to the fire (so to speak) and further confirmed and strengthened our resolute determination to unravel the too-long hidden mysteries of philosophy and science contained in ancient India's Vedic lore, with the consequence that, after eight years of concentrated contemplation in forest-solitude, we were at long last able to recover the long lost keya which alone could unlock the portals thereof.

9. And we were agreeably astonished and intensely gratified to find that exceedingly tough mathematical problems (which the mathematically most advanced present day Western scientific world had spent huge lots of time, energy and money on and which even now it solves with the utmost difficulty and after vast labour involving large numbers of difficult, tedious and cumbeisome "steps" of working) can be easily and readily solved with the help of these ultra-easy Vedic Siitras (or mathematical aphorisms) contained in the Pariśiṣṭa (the Appendix-portion) of the ATHARVAVEDA in a few simple steps and by methods which can be conscientiously described as mere "mental arithmetic".

10. Eversince (i.e. since several decades ago), we have been carrying on an incessant and strenuous campaign for the India-wide diffusion of all this scientific knowledge, by means of lectures, blackboard- demonstrations, regular classes and so on in schools, colleges, universities etc., all over the country and have been astounding our audiences everywhere with the wonders and marvels not to say, miracles of Indian Vedic mathematics.

11. We were thus at last enabled to succeed in attracting the more than passing attention of the authorities of several Indian universities to this subject. And, in 1962, the Nagpur University not merely had a few lectures and blackboard-demonstrations given but also arranged for our holding regular classes in Vedir mathematics (in the University's Convocation

Hall) for the benefit of all in general and especially of the University and college professors of mathematics, physics etc.

12. And, consequently, the educationists and the cream of the English educated section of the people including the highest officials (e.g. the high-court judges, the ministers etc.,) and the general public as such were all highly impressed ; nay, thrilled, wonder-struck and flabbergasted! And not only the newspapers but even the University's official reports described the tremendous sensation caused thereby in superlatively eulogistic terms ; and the papers began to refer to us as "the Octogenarian Jagadguru Shankaracharya who had taken Nagpur by storm with his Vedic mathematics", and so on !

13. It is manifestly impospible, in the course of a short note [in the nature of a "trailer"), to give a full, detailed, thorough-going, comprehensive and exhaustive description of the unique features and startling characteristics of all the mathematical lore in question. This can and will be done in the subsequent volumes of this series (dealing seriatim and in extenso with all the various portions of all the various branches of mathematics).

14. We may, however, at this point, draw the earnest attention of every one concerned to the following salient items thereof :—

(i) The Siitras (aphorisms) apply to and cover each and every part of each and every chapter of each and every branch of mathematics (including arithmetic, algebra. geometry—plane and solid, trigonometry—plane and spherical, conics—geometrical and analytical, astronomy, calculus—differential and integral etc., etc. In fact, there is no part of mathematics, pure or applied, which is beyond their jurisdiction ;

(ii) The Sūtras are easy to understand, easy to apply and easy to remember ; and the whole work can be truthfully summarised in one word "mental" !

(iii) Even as regards complex problems involving a good number of mathematical operations (consecutively or even simultaneously to be performed), the time taken by the Vedic method will be a third, a fourth, a tenth or even a much smaller fraction of the time required according to modern (i.e. current) Western methods;

(iv) And, in some very important and striking cases, sums requiring 30, 50, 100 or even more numerous and cumbrous "steps" of working (according to the current Western methods) can be answered in a single and simple step of work by the Vedic method! And little children (of only 10 or 12 years of age) merely look at the sums written on the blackboard (on the platform) and immediately shout out and dictate the answers from the body of the convocation hall (or other venue of the demonstration). And this is because, as a matter of fact, each digit automatically yields its predecessor and its successor! and the children have merely to go on tossing off (or reeling off) the digits one after another (forwards or backwards) by mere mental arithmetic (without needing pen or pencil, paper or slate etc)!

(v) On seeing this kind of work actually being performed by the little children, the doctors, professors and other "big-guns" of mathematics are wonder struck and exclaim :—"Is this mathematics or magic"? And we invariably answer and say: "It is both. It is magic until you understand it; and it is mathematics thereafter"; and then we proceed to substantiate and prove the correctness of this reply of ours! And

(vi) As regards the time required by the students for mastering the whole course of Vedic mathematics as applied to all its branches, we need merely state from our actual experience that 8 months (or 12 months) at an average rate of 2 or 3 hours per day

should suffice for completing the whole course of mathematical studies on these Vedic lines instead of 16 or 20 years required according to the existing systems of the Indian and also of foreign universities.

15. In this connection, it is a gratifying fact that unlike some so-called Indologists (of the type hereinabove referred to) there have been some great modern mathematicians and historians of mathematics (like Prof. G. P. Halstead, Professor Ginsburg, Prof. De Moregan, Prof. Hutton etc.,) who have, as truth-seekers and truth-lovers, evinced a truly scientific attitude and frankly expressed their intense and whole-hearted appreciation of ancient India's grand and glorious contributions to the progress of mathematical knowledge (in the Western hemisphere and elsewhere).

16. The following few excerpts from the published writings of some universally acknowledged authorities in the domain of the history of mathematics, will speak eloquently for themselves:—

(i) On page 20 of his book "On the Foundation and Technique of Arithmetic", we find Prof. G.P. Halstead saying "The importance of the creation of the ZERO mark can never be exaggerated. This giving of airy nothing not merely a local habitation and a name, a picture but helpful power is the characteristic of the Hindu race whence it sprang. It is like coining the Nirvāṇa into dynamos. No single mathematical creation has been more potent for the general on-go of intelligence and power".

(ii) In this connection, in his splendid treatise on "The present mode of expressing numbers" (the Indian Historical Quarterly Vol. 3, pages 530-540) B. B. Dutta says: "The Hindus adopted the decimal scale vary early. The nuinerical language of no other nation is so scientific and has attained as high a state of perfection as that of the ancient Hindus.

In symbolism they succeeded with ten signs to express any number most elegantly and simply. It is this beauty of the Hindu numerical notation which attracted the attention of all the civilised peoples of the world and charmed them to adopt it"

(iii) In this very context, Prof. Ginsburg says:—

"The Hindu notation was carried to Arabia about. 770 A.D. by a Hindu scholar named KAṄKA who was invited from Ujjain to the famous Court of Baghdad by the Abbaside Khalif AL-MANSUR. Kaṅka taught Hindu astronomy and mathematics to the Arabian scholars ; and, with his help, they translated into Arabic the Brahma-Sphuṭa-Siddhānta of Brahma Gupta. The recent discovery by the French savant M.F. NAU proves that the Hindu numerals were well known and much appreciated in Syria about the middle of the 7th Century A-D". (GINSBURG'S "NEW LIGHT on our numerals", Bulletin of the American Mathematical Society, Second series, Vol. 25, pages 366-369).

(iv) On this point, we find B. B. Dutta further saying:

"From Arabia, the numerals slowly marched towards the West through Egypt and Northern Arabia; and they finally entered Europe in the 11th Century. The Europeans called them the Arabic notations, because they received them from the Arabs. But the Arabs themselves, the Eastern a6 well as the Western, have unanimously called them the Hindu figures. (Al-Arqan-Al-Hindu".)

17. The above-cited passages are, however, in connection with and in appreciation of India's invention of the "ZERO" mark and her contributions of the 7th century A.D. and later to world mathematical knowledge.

In the light, however, of the hereinabove given detailed description of the unique merits and characteristic excellences of the still earlier Vedic Sūtras dealt with in the 16 volumes of

this series[1], the conscientious (truth-loving and truth-telling) historians of Mathematics (of the lofty eminence of Prof. De Morgan etc.) have not been guilty of even the least exaggeration in their candid admission that "even the highest and farthest reaches of modern Western mathematics have not yet brought the Western world even to the threshold of Ancient Indian Vedic Mathematics".

18. It is our earnest aim and aspiration, in these 16 volumes[1], to explain and expound the contents of the Vedic mathematical Siitras and bring them within the easy intellectual reach of every seeker after mathematical knowledge.

B.—EXPLANATORY EXPOSITION OF SOME SALIENT, INSTRUCTIVE AND INTERESTING ILLUSTRATIVE SAMPLE SPECIMENS BY WAY OF COMPARISON AND CONTRAST

Preliminary Note :—

With regard to every subject dealt with in the Vedic Mathematical Siitras, the rule generally holds good that the Siitras have always provided for what may be termed the *'General Case'* (by means of simple processes which can be easily and readily—nay, instantaneously applied to any and every question which can possibly arise under any particular heading.

2. But, at the same time, we often come across special cases which, although classifiable under the general heading in question, yet present certain additional and typical characterestics which render them still easier to solve. And, therefore, special provision is found to have been made for such special cases by means of special Siitras, sub-Sūtras, corollaries etc., relating and applicable to those particular types alone.

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[1] **Only one volume has been bequeathed by His Holiness to posterity** cf **p. x above—General** ***Editor.***

3. And all that the student of these Siitras has to do is to look for the special characteristics in question, recognisc the particular type before him and determine and npply the special formula prescribed therefor.

4. And, generally speaking it is only in *case* no special case is involved, that the general formula has to be resorted to. And this process is naturally a little longer. But it need hardly be pointed out that, even then, the longest of the methods according to the Vedic system comes nowhere (in respect of length, cumbrousness and tediousness etc.,) near the corresponding process according to the system now current everywhere.

5. For instance, the conversion of a vulgar fraction (say $\frac{1}{19}$ or $\frac{1}{29}$ or $\frac{1}{49}$ etc.,) to its equivalent recurring decimal shape involves 18 or 28 or 42 or more steps of cumbrous working (according to the current system) but requires only one single and simple step of mental working (according to the Vedic Sūtras) !

6. This is not all. There are still other methods and processes (in the latter system) whereby even that very small (mental) working can be rendered shorter still ! This and herein is the beatific beauty of the whole scheme.

7. To start with, we should naturally have liked to begin this explanatory and illustrative exposition with a few processes in arithmetical computations relating to multiplications and divisions of huge numbers by big multipliers and big divisors respectively and then go on to other branches of mathematical calculation.

8. Rnt, as we have *just hereinabove* referred to a particular bnt wonderful type of mathematiral work wherein 18, 28, 42 or even more steps of working can be condensed into a single-step answer which can be written down immediately (by means of what we have been describing as straight, single-line, mental arithmetic) ; and, as this statement must naturally have aroused intense eagerness and curiosity in the minds of the students (and the teachers too) and especially as the process is

based on elementary and basic fundamental principles and no previous knowledge of anything in the nature of an indispensable and inescapable pre-requisite ohapter, subject and so on, we are beginning this exposition here with an easy explanation and a simple elucidation of that particular illustrative specimen.

9. And then we shall take up the other various parts, one by one, of the various branches of mathematical computation and hope to throw sufficient light thereon to enable the students to make their own comparison and contrast and arrive at correct conclusions on all the various points dealt with.

C. ILLUSTRATIVE SPECIMEN SAMPLES

(Comparison and Contrast)

SAMPLE SPECIMENS

OF

ARITHMETICAL COMPUTATIONS

I. *Multiplication*: *The "Sanskrit Sūtra" (Formula) is*—

(i) *Multiply 87265 by 32117* ॥ ऊर्ध्वतिर्यग्भ्याम् ॥

By *current method*:

```
     87265
     32117
    ------
    610855
    87285
   87265
  174530
 261795
----------
2802690005
----------
```

By Vedic *mental one-line method*:

```
     87285
     32117
----------
2802690005
----------
```

Note: Only the answer is written automatically down by *Ūrdhwa Tiryak Sara* (forwards or backwards).

II. *Division* :

(2) Express $\frac{1}{19}$ in its full recurring decimal shape (18 digits) :

By the current method :

19) 1·00(·052631578947368421̇

95
——
50
38
——
120
114
——
60
57
——
30
19
——
110
95
——
150
133
——
170
152
——
180
171
——
00
76
——
140
133
——
70
57
——
130
114
——
160
152
——
80
76
——
40
38
——
20
19
——
1

The "Sanskrit Sūtra" (Formula) is ;

॥ एकाधिकेन पूर्वेण ॥

By the Vedic mental one-line method :

(by the Ekādhika-Pūrva Sūtra)

(forwards or backwards), we merely write down the 18-digit-answer :—

·0̇52631578
947368421̇

Division continued:

Note: $\frac{1}{49}$ gives 42 recurring decimal places in the answer but these too are written down mechanically in the same way (backwards or forwards). And the same is the case with all such divisions (whatever the number of digits may be):

(3) *Divide 7031985 by 823*:

By the current method: *By the mental Vedic one-line method*:

```
823)7031985(8544          8123)70319(85
     6584                      675
     ----                  ----------
      4479                 8544(273
      4115                 ----------
      ----
       3648
       3292
       ----
        3565
        3292
        ----
         273
        ----
   ∴ Q=8544
     R= 273
```

(4) Divide .0003147 by 814266321 (to 6 decimal places):

The current method is notoriously too long, tedious, cumbrous and clumsy and entails the expenditure of enormous time and toil. Only the Vedic mental one-line method is given here. The truth-loving student can work it out by the other method and compare the two for himself.

```
8/1425632)·00034147
         )     3295
         ----------
          ·0000419...
```

(5) *Find the Reciprocal of 7246041 to eleven Decimal places*:

By the Vedic mental one-line-method.

(by the *Ūrdhwa-Tiryak Sūtra*)

```
7/246041)·000001000000
                374610
         -------------
          ·00000013800...
         -------------
```

N.B.:—The same method can be used for 200 or more places.

III. *Divisibility* :

(6) ***Find out whether 5293240096 is divisible by 139*** :

By the current method, nothing less than complete division will give a clue to the answer (Yes or No).

But by the Vedic mental one-line method (by the ***Ekādhika-Pūrva Sūtra***), we can at once say :—

for)	5	2	9	3	2	4	0	0	9	6
139)	139	89	36	131	29	131	19	51	93	

} ∴ YES.

IV. *Square Root* :

(7) *Extract the square root of 738915489* :

By the current method :

```
  738915489(27183
  4
  ---
 47)338
    329
    ---
541)  991
      541
      ---
5428) 45054
      43424
      -----
54363) 163089
       163089
       ------
            0
```

By the Vedic mental one-line method:

```
4)738915489
   35613674
   --------
   27183.000 Ans.
```

(By the ***Ūrdhwa-Tiryak Sūtra***)

∴ The square root is **27183.**

(8) ***Extract the square root of 19.706412814 to 6 decimal places*** :

The current method is too cumbrous and may be tried by the student himself.

The Vedic mental one-line method (by ***Ūrdhwa-Tiryak Sūtra***) is as follows :—

```
8)19·706412814
     ·351010151713
  ----------------
   4·439190 . . .
```

V. Cubing and Cube-Root: The "Sanskrit *Sūtra*" (*Formula*) is :—

(9) Find the cube of 9989. ॥ ~ I T कृावदूनीकृत्य वर्गं च योजयेत् ॥

The current method is too cumbrous.

The Vedic mental one-line method (by the *Yāvadūnam-Tāvadūnam* Sūtra) is as follows:—

$$9989^3 = 9967/0363/\overline{1331} = 9967/0362/8669$$

(10) *Extract* the Cube-Rod of *355045312441:*

The current method is too cumbrous.

The Vedic mental one-line method is as follows:—

$$\sqrt[3]{355045312441} = 7\ .\ .\ 1 = 7081$$

SAMPLE SPECIMENS FROM ALGEBRA

I. *Simple Equations*: The "Sanskrit *Sūtra*" (Formula) is :—

(11) Solve: $\frac{3x+4}{6x+7} = \frac{x+1}{2x+3}$ ॥ शून्यं साम्यसमुच्चये ॥

By the current method:

$\therefore 6x^2+17x+12 = 6x^2+13x+7$

$\therefore 4x = -5$

$\therefore x = -1\frac{1}{4}$

By the Vedic method (by the *Śūnyam-Samuccaya Sūtra*

$\therefore 4x+5=0 \ \therefore x = -1\frac{1}{4}$

(12) $\frac{4x+21}{x+5} + \frac{5x-69}{x-14} = \frac{3x-5}{x-2} + \frac{6x-41}{x-7}$

The current method is too cumbrous.

The Vedic method simply says: $2x-9=0 \ \therefore x = 4\frac{1}{2}$

(13) $\left(\frac{x-5}{x-7}\right)^3 = \frac{x-3}{x-9}$

The current niethod is horribly cumbrous.

The Vedic method simply says: $4x-24=0 \ \therefore x=6.$

11. Quadratic-Equations (and Calculus):

The same is the case here.

(14) $\frac{16x-3}{7x+7}=\frac{2x-15}{11x-25}$ $\therefore$ x=1 or 10/9

(15) $\frac{3}{x+3}+\frac{4}{x+4}=\frac{2}{x+2}+\frac{5}{x+5}$ $\therefore$ x=0 or −7/2.

(16) $7x^2-11x-7=0$

By Vedic method (by *"Calana-kalana"* **Sūtra**; Formula) i.e., by Calculus-Formula we sap: $14x-11=\pm\sqrt{317}$.

N.B. :—Every quadratic can thus be broken down into two binomial-factors. And the same principle can be utilised for cubic, biquadratic, pentic etc., expressions.

III. *Summation* of **Series**:

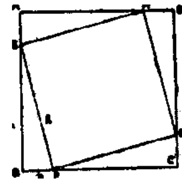
The current methods are horribly cumbrous. The Vedic mental one-line methods are very simple and easy.

(17) $\frac{1}{56}+\frac{1}{72}+\frac{1}{90}+\frac{1}{110}=4/77$

(18) $\frac{3}{70}+\frac{9}{190}+\frac{27}{874}+\frac{99}{8670}=\frac{138}{1015}$

SPECIMEN SAMPLES FROM GEOMETRY

(19) Pythogoras Theorem is constantly required in all mathematical work, but the proof of it is ultra-notorious for its cumbrousness, clumsiness, etc. There are several Vedic proofs thereof (every one of them much simpler than *Euclid's*). I give two of them below:—



E, F, G and H are points on AB, BC, CD and DA such that AE=BF=CG=DH. Thus ABCD is split up into the square EFGH and 4 congruent triangles.

Their total area $=H^2+4\times\frac{1}{2}\times mn$

$\rightarrow(H^2+4\times\frac{1}{2}mn)$

$=H^2+2mn$

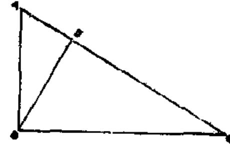
But the area of ABCD is $(m+n)^2$

$=m^2+2mn+n^2$

$\therefore H^2+2mn=m^2+2mn+n^2$

$\therefore H^2=m^2+n^2$. Q.E.D.

(20) *Second* Proof:



Drew BD ⊥ to AC.

Then ABC, ADB end BDC are similar.

$$\frac{ADB}{ABC}=\frac{AB^2}{AC^2} \text{ and } \frac{BDC}{ABC}=\frac{BC^2}{AC^2}$$

$$\therefore \frac{ADB+BDC}{ABC}=\frac{AB^2+BC^2}{AC^2}.$$ But ; ADB+BDC=ABC

$\therefore AB^2+BC^2=AC^2$. Q.E.D.

Note:—Apollonius Theorem, Ptolemy's Theorem, ctc., etc., are all similarly proved by very simple and easy methods.

SIMPLE SPECIMENS FROM CONICS AND CALCULUS

(21) Equation of the straight line joining two points:

For finding the equation of the straight line passing through two points (whose co-ordinates are given).

Say (9, 17) and (7, —2).

By the *Current* Method:

Let the equation be $y=mx+c$.

$\therefore 9m+c=17$; and $7m+c=-2$

Solving this simultaneous-equation in m and c.

We have $2m=19$; $\therefore m=9\frac{1}{2}$ $\therefore C=-68\frac{1}{2}$

Substituting ; these, values, we have $y=9\frac{1}{2}x-68\frac{1}{2}$

$\therefore 2y=19x-137$ $\therefore$ *19x—2y=137.* But this method is cumbrous.

Second method using the formula $y-y^1=\frac{y^{11}-y^1}{x^{11}-x^1}(x-x^1)$ is still more cumbrous (and confusing),

But the Vedic *mental* one-line method by the Sanskrit Sūtra (Formula), ॥ परावर्त्य योजयेत् ॥ ("*Parāvartya*-Sūtra") enables us to write down the answer *by* a mere look at the given co-ordinates.

(22) ***When*** does a ***general-equation*** represent ***two*** straight lines ?

Say, $12x^2+7xy-10y^2+13x+45y-35=0$

By the Current Method.

Prof. S. L. Loney devotes about 15 lines (section 119, Ex. 1 on page 97 of his "Elements of Co-ordinate Geometry") to hie. "model" solution of this problem as follows:—

Here $a=12$, $h=7/2$, $b=-10$, $g=13/2$, $f=45/2$ and $c=-35$.

$\therefore abc+2fgh-af^2-bg^2-ch^2$ turns out to be zero.

$$=12(x-10)(x-35)+\frac{2\times 45}{2}\times\frac{13}{2}\times\frac{7}{2}-\frac{12(45)^2}{2}-(-10)\frac{(13)^2}{2}$$

$$-(-35)\frac{(7)^2}{2}=4200+\frac{4095}{4}-6075+\frac{1690}{4}+\frac{1715}{4}=-1875+\frac{7500}{4}=0$$

The equation represents two straight lines.
Solving it for x, we have :—

$$x^2+\frac{7y+13}{12}+\left(\frac{7y+13}{24}\right)^2=\frac{10y^2-45y+35}{12}+\left(\frac{7y+13}{24}\right)^2$$

$$=\left(\frac{23y-43}{24}\right)^2$$

$$\therefore x+\frac{7y+13}{24}=\frac{23y-43}{24}$$

$$\therefore x=\frac{2y-7}{3} \text{ or } -\frac{5y+5}{4}$$

$\therefore$ The two straight lines are $3x=2y-7$ and $4x=-5y+5$.

By the Vedic *method,* however, we at once apply the "*Ādyam-ādyena*" Sūtra and (by merely looking at the quadratic) write down the answer:

Yes; and the straight lines are $3x-2y=-7$ and $4x+5y=5$.

(23) Dealing with the same principle and adopting the same procedure with regard to hyperbolas, conjugate hyperbolas and asymptotes, in articles 324 and 325 on pages 293 and 294 of his "Elements of Co-ordinate Geometry" Prof. S. L. Loney devotes 27+14(=41 lines) to the problem and says:—

As $3x^2-5xy-2y^2+5x-11y-8=0$ is the equation to the given hyperbola.

$\therefore 3(-2)c+2.\frac{5}{2}.\frac{11}{2}(-\frac{5}{2})-3(\frac{11}{2})^2-(-2)(\frac{5}{2})^2-c(-\frac{5}{2})^2=0.$

$\therefore c=-12.$

$\therefore$ The equation to the asymptotes is $3x^2-5xy-2y^2+5x-11y-12=0$

and the equation to the conjugate-hyperbola is $3x^2-5xy-2y^2+5x+15y-16=0$

By the Vedic method, however, we use the same (*'Ādyam-ādyena'*) Sūtra and automatically write down the equation to the asymptotes and the equation to the conjugate-hyperbola.

The Vedic methods are so simple that their very simplicity is astounding, and, as Desmond Doig has aptly, remarked, it is *difficult* for any one to believe it until one *actually* sees it.

It will be our aim in this and the succeeding volumes[1] to bring this long-bidden treasure-trove of mnthemetical knowledge within easy reach of everyone who wishes to obtain it and benefit by it.

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1 This is the only volume left by the author—Editor.

TEXT

॥ ॐ श्रीः ॥

VEDIC MATHEMATICS

OR

SIXTEEN SIMPLE MATHEMATICAL FORMULAE FROM THE VEDAS

SIXTEEN SŪTRAS AND THEIR COROLLARIES

Sūtras	*Sub-Sūtras or Corollaries*
1. एकाधिकेन पूर्वेण *Ekādhikena Pūrveṇa* (also a corollary)	1. आनुरूप्येण *Ānurūpyeṇa*
2. निखिलं नवतश्चरमं दशतः *Nikhilaṃ Navataścaramap Daśataḥ*	2. शिष्यते शेषसंज्ञः *Śiṣyate Śeṣasaṃjñaḥ*
3. ऊर्ध्वतिर्यग्भ्याम् *Ūrdhva-tiryagbhyāṃ*	3. आद्यमाद्येनान्त्यमन्त्येन *Ādyamādyenāntya-mantyena*
4. परावर्त्य योजयेत् *Parāvartya Yojayet*	4. केवलैः सप्तकं गुण्यात् *Kevalaiḥ Saptakaṃ Guṇyit*
5. शून्यं साम्यसमुच्चये *Śūnyaṃ Sāmyasamuccaye*	6. वेष्टनम् *Veṣṭanaṃ*
6. (आनुरूप्ये) शून्यमन्यत् (*Ānurūpye*) *Śūnyamanyat*	6. यावदूनं तावदूनम् *Yāvadūnaṃ Tāvadūnaṃ*
7. संकलनव्यवकलनाभ्याम् *Saṅkalana-vyavakalanābhyām* (*also* a corollary)	7. यावदूनं तावदूनीकृत्य वर्गं च योजयेत् *Yāvadunaṃ Tāvadūnīkṛtya Vargañca Yojayet*
8. पूरणापूरणाभ्याम् *Pūraṇāpūraṇābhyām*	8. अन्त्ययोर्दशकेऽपि *Antyayordaśake'pi*
9. चलनकलनाभ्याम् *Calana-Kalanābhyām*	9. अन्त्ययोरेव *Antyayoreva*

Sūtras

10. यावदूनम्
Yāvadūnam

11. व्यष्टिसमष्टिः
Vyaṣṭisamaṣṭiḥ

12. शेषाण्यङ्केन चरमेण
Śeṣāṇyaṅkena Carameṇa

13. सोपान्त्यद्वयमन्त्यम्
Sopāntyadvayamantyam

14. एकन्यूनेन पूर्वेण
Ekanyūnena Pūrveṇa

15. गुणितसमुच्चयः
Guṇitasamuccayaḥ

16. गुणकसमुच्चयः
Guṇakasamuccayaḥ

Sub-Sūtras or Corollaries

10. समुच्चयगुणितः
Samuccayaguṇitaḥ

11. लोपनस्थापनाभ्याम्
Lopanasthāpanābhyām

12. विलोकनम्
Vilokanam

13. गुणितसमुच्चयः समुच्चयगुणितः
Gunitauuccayaḥ
Samuccayaguṇitaḥ

[Note—This list has been compiled from stray references in the text—EDITOR.]

॥ ॐ श्रीः ॥

PROLEGOMENA

In our "Descriptive, Prefatory Note on the Astounding Wonders of Ancient Indian Vedic Mathematics", we have again and again, so often and at such great length and with such wealth of detail, dwelt on the almost incredible simplicity of the Vedic Mathematical Sūtras (aphorisms or formulae) and the indescribable ease with which they can be understood, remembered and applied (even by little children) for the solution of the wrongly-believed-to-be-"difficult"problems in the various branches of Mathematics, that we need not, at this point, traverse the same ground and cover the same field once again here.

Suffice it, for our present immediate purpose, to draw the earnest attention of every scientifically-inclined mind and researchward-attuned intellect, to the remarkably extra-ordinary and characteristic—nay, unique fact that the Vedic system does not academically countenance (or actually follow) any automatical or mechanical rule even in respect of the correct sequence or order to be observed with regard to the various subjects dealt with in the various branches of Mathematics (pure and applied) but leaves it entirely to the convenience and the inclination, the option, the temperamental predilection and even the individual idiosyncracy of the teachers and even the students themselves (as to what particular order or sequence they should actually adopt and follow)!

This manifestly out-of-the-common procedure must doubtless have been due to some special kind of historical back-ground, background which made such a consequence not only natural but also inevitable under the circumstances in question.

Immemorial tradition has it and historical research confirms the orthodox belief that the Sages, Seers and Saints of ancient India (who are accredited with having observed, studied

and meditated in the Aranya (i.e. in **forest-solicitude)—on** physical Nature around them and deduced their grand ***Vedantic Philosophy*** therefrom as the result not only of their theoretical reasonings but also of what may be more fittingly described as True Realisation by means of Actual VISUALISATION) seem to have similarly observed, studied and meditated on the mysterious workings of numbers, figures etc. of the mathematical world (to wit, Nature) around them and deduced their ***Mathematical*** Philosophy therefrom by a similar process of what one may, equally correctly, describe as processes of True-Realisation by means of Actual VISUALISATION.

And, consequently, it naturally follows that, in-as-much as, unlike human beings who have their own personal prejudices, partialities, hatreds and other such ***subjective*** factors distorting their visions, warping their judgements and thereby contributing to their inconsistent or self-contradictory decisions and discriminatory attitudes, conducts etc.), ***numbers*** (in **Mathematics)** labour under no such handicaps and disadvantages based on personal prejudices, partialities, hatreds etc. They are, on the contrary, strictly and purely ***impersonal*** and ***objective*** in their behaviour **etc.,** follow the same rules uniformly, **consistenly** and **invariatly** (with no question of outlook, approach, personal psychology etc. involved therein) and are therefore **absolutely** reliable and dependable.

This seems to have been the real historical reason why. barring a few unavoidable exceptions **in** the shape of elementary, basic and fundamental first principles (of a preliminary or pre-requisite character), almost all the subjects dealt with in the various branches of Vedic Mathematics are explicable and expoundable on the basis of those very 'basic principles' or 'first principles', with the natural consequence that no particular subject or **subjects** (or chapter or chapters) need necessarily precede or follow some other particular subject or subjects (or chapter or chapters).

Nevertheless, it is also undeniable that, although ***any*** particular sequence is quite possible, permissible and feasible

yet, some particular sequence will actually have to be adopted by a teacher (and, much more therefore, by an author). And so, we find that subjects like analytical conics and even calculus (differential and integral) (which is usually the bugbear and terror of even the advanced students of mathematics under the present system all the world over) are found to figure and fit in at a very early stage in our Vedic Mathematics (because of their being expounded and worked out on basic first principles. And they help thereby to facilitate mathematical study especially for the small children).

And, with our more-than-half-a-century's actual personal experience of the very young mathematics-students and their difficulties, we have found the Vedic sequence of subjects and chapters the most suitable for our purpose (namely, the eliminating from the children's minds of all fear and hatred of mathematics and the implanting therein of a positive feeling of exuberant love and enjoyment thereof)! And we fervently hope and trust that other teachers too will have a similar experience and will find us justified in our ambitious description of this volume as "Mathematics without tears".

From the herein-above described historical back-ground to our Vedic Mathematics, it is also obvious that, being based on basic and fundamental principles, this system of mathematical study cannot possibly come into conflict with any other branch, department or instrument of science and scientific education. In fact, this is the exact reason why all the other sciences have different *Theories* to propound but Mathematics has only THOEREMS to expound!

And, above all, we have our Scriptures categroically laying down the wholesome dictum :—

युक्तियुक्तं वचो ग्राह्यं बालादपि शुकादपि ।
युक्तिहीनं वचस्त्याज्यं वृद्धादपि शुकादपि ॥

(i.e. whatever is consistent with right reasoning should be accepted, even though it comes from a boy or even from a parrot; and whatever is inconsistent therewith ought to be

rejected, although emanating from an old man or even from the great sage Shree Shuka himself.

In other words, we are called upon to enter on such a scientific quest as this, by divesting our minds of all pre-conceived notions, keeping our minds ever open and, in all humility (as humility alone behoves and befits the real seeker after truth), welcoming the light of knowledge from whatever direction it may be forthcoming. Nay, our scriptures go so far as to inculcate that even thir expositions should be looked upon by us not as "teachings" or even as advice, guidance etc. but as acts of "thinking aloud" by a fellow student.

It is in this spirit and from this view-point that we now address ourselves to the task before us, in this series of volumes[1] (i.e. a sincere exposition of the mathematical Sūtras under discussion, with what we may call our "running comments" (just as in a blackboard demonstration or a magic lantern lecture or a cricket match etc. etc.).

In conclusion, we appeal to our readers (as we always, appeal to our hearers) to respond hereto from the same stand-point and in the same spirit as we have just hereinabove described.

We may also add that, inasmuch as we have since long promised to make these volumes[2] "self-contained", we shall make our explanations and expositions as full and clear as possible. Brevity may be the soul of wit; but certainly not at the expense of CLARITY (and especially in mathematical treatises like these).

॥ ॐ तत् सत् ॥

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[1], [2] Unfortunately, only one volume has been left over by His Holiness. —Editor.

॥ॐ श्रीः॥

ACTUAL APPLICATIONS

OF

THE VEDIC SŪTRAS

TO

CONCRETE MATHEMATICAL PROBLEMS

CHAPTER I

A SPECTACULAR ILLUSTRATION

For the reasons just explained immediately hereinbefore let us take the question of the **CONVERSION** of Vulgar fractions into their equivalent decimal form.

First *Example* :

Case 1. 1/19 } And there, let us first deal with the case of a fraction (say 1/19) whose denominator ends in 9.

By the Current Method.

19)1.00(.0 5 2 6 3 1 5 7 8 9 4 7 3 6 8 4 2 $\dot{1}$
95

50	170	
38	152	
120	180	
114	171	
60	90	160
57	76	152
30	140	
19	133	
110	70	40
95	57	38
150	130	20
133	114	19
170	160	1

By the Vedic one-line mental method.

A. *First method.*

$\frac{1}{19} = .\dot{0}\ 5\ 2\ 6\ 3\ 1\ 5\ 7\ 8$
1 1 1 1 1 1
9 4 7 3 6 8 4 2 $\dot{1}$
1 1 1

B. *Second method.*

$\frac{1}{19} = .\dot{0}\ 5\ 2\ 6\ 3\ 1\ 5\ 7\ 8/9\ 4\ 7\ 3\ 6\ 8\ 4\ 2\ \dot{1}$
1 1 1 1 1 1/ 1 1 1

This is the whole working. And the *modus operandi* is explained in the next few pages.

It is thus apparent that the 18-digit recurring-decimal answer requires 18 steps of working according to the current system but only one by the Vedic Method.

Explanation :
The relevant Sūtra reads : एकाधिकेन पूर्वेण (Ekādhikena Pūrveṇa) which, rendered into English, simply says : "By one more than the previous one". Its application and *modus operandi* are as follows :—

(i) The last digit of the denominator in this case being 1 and the previous one being 1, "one more than the previous one" evidently means 2.

(ii) And the preposition "by" (in the Sfitra) indicates that the arithmetical operation prescribed is either multiplication or division. For, in the case of addition and subtraction, to and *from* (respectively) would have been the appropriate preposition to use. But "by" is the preposition actually found used in the Sfitra. The inference is therefore obvious that either multiplication or division must be enjoined. And, as both the meanings are perfectly correct and equally tenable (according to grammar and literary usage) and as there is no reason—in or from the text—for one of the meanings being accepted and the other one rejected, it further follows that both the processes are actually meant. And, as a matter of fact, each of them actually serves the purpose of the Siitra and fits right into it (as we shall presently show, in the immediately following explanation of the *modus operandi* which enables us to arrive at the right answer by either operation).

A. The *First* method:

The first method is by means of multiplication by 2 (which is the "Ekādhika Pūrva" i.e. the number which is just one more than the penultimate digit in this case).

Here, for reasons which will become clear presently, we can know beforehand that the last digit of the answer is bound to be 1! For, the relevant rule hereon (which wc shall explain and expound at a later stage) stipulates that the product of the last digit of the denominator and the last digit of the decimal equivalent of the fraction in question must invariablv end in 9. Therefore, as the last digit of the denominator in this case is 9, it automatically follows that the last digit of the decimal equivalent is bound to be 1 (so that the product of the multiplicand and the multiplier concerned may end in 9).

We, therefore, start with 1 as the last (i.e. the right-hand-most) digit of the answer and proceed leftward continuously multiplying by 2 (which is the Ekādhika Pūrva i.e. one more than the penultimate digit of the denominator in this case) until a repetition of the whole operation stares us in the face and intimates to us that we are dealing with a Recurring Decimal and may therefore put up the usual recurring marks (dots) and stop further multiplication-work.

Our *modus-operandi*-chart is thus as follows:—

(i) We put down 1 as the right-hand most digit	1
(ii) We multiply that last digit (1) by 2 and put the 2 down as the immediately preceding digit	2 1
(iii) We multiply that 2 by 2 and put 4 down as the next previous digit	4 2 1
(iv) We multiply that 4 by 2 and put it down, thus	8 4 2 1
(v) We multiply that A by 2 and get 16 as the product. But this has two digits. We therefore put the 6 down immediately to the left of the 8 and keep the 1 on hand to be carried over to the left at the next step (as we always do in all multiplication e.g. of 69 X 2 = 138 and so on).	6 8 4 2 1 1

(vi) We now multiply the 6 by 2, get 12 as the product, add thereto the **1** (kept to be carried over from the right at the last step), get **13** as the consolidated product, put the 3 down and keep the 1 on hand for carrying over to the left at tho next step.

```
 3 6 8 4 2 1
1 1
```

(vii) We then multiply the **3** by 2, add the one carried over from the right one, get **7** as the consolidated product. But, as this is a single-digit number (with nothing to carry over to the left), we put it down as our next multiplicand.

```
7 3 6 8 4 2 1
 1 1
```

(viii-xviii) We follow this procedure continually until we reach the 18th digit (counting leftwards from the right), when we find that the whole decimal has begun to repeat itself. We therefore put up the usual recurring marks (dots) on the first and the last digits of the answer (for betokening that the whole of it is a Recurring Decimal) and stop the multiplication there.

Our chart now reads as follows :—

```
1/19 = . 0 5 2 6 3 1 5 7 8/9 4 7 3 6 8 4 2 1
           1   1     1 1 1 1/ 1   1 1
```

We thus find that this answer obtained by us with the aid of our Vedic one-line mental arithmetic is just exactly the same as we obtained by the current method (with its 18 steps of Division-work).

In passing, we may also just mention that the current process not only takes 18 steps of working for getting the 18 digits of the answer not to talk of the time, the energy, the paper, the ink etc. consumed but also suffers under the

additional and still more serious handicap that, at each step, a probable "trial" digit of the Quotient has to be taken on ***trial*** for multiplying the divisor which, is sometimes found to have played on us the scurvy trick of yielding a product larger than the dividend on hand and has thus — after trial — to be discarded in favour of another "trial" digit and so on. In the Vedic method just above propounded, however, there are no subtractions at all and no need for such trials, experiments etc., and no scope for any tricks, pranks and so on but only a straightforward multiplication of single-digit numbers; and the multiplier is not merely a simple one but also the same throughout each particular operation. All this lightens, facilitates and expedites the work and turns the study of mathematics from a burden and a bore into a thing of beauty and a joy for ever (so far, at any rate, as the children are concerned).

In this context, it must also be ***transparently*** clear that the long, tedious, cumbrous and clumsy methods of the current system tend to afford greater and greater scope for the children's making of mistakes (in the course of all the long multiplications, subtractions etc. involved therein); and once one figure goes wrong, the rest of the work must inevitably turn out to be an utter waste of time, energy and so on and engender feelings of fear, hatred and disgust in the children's minds.

***B. The Second method*:**

As already indicated, the second method is of division (instead of multiplication) by .the self-same **"Ekādhika Pūrva"**, namely 2. And, as division is the exact opposite of multiplication, it stands to reason that the operation of division should proceed, not from right to left (as in the case of multiplication as expounded hereinbefore) but in the exactly opposite direction (i.e. from left to right). And such is actually found to be the case. Its application and ***modus operandi*** are as follows:—

(i) Dividing 1 (the first digit of the dividend) by 2, we see the quotient is zero and the remainder is 1. We, therefore, set 0 down as the first digit of the quotient and ***prefix*** the Remainder (1) to that very digit of the Quotient (as a sort of reverse-procedure to the

carrying-to-the-left process used in multiplication) and thus obtain 10 as our next Dividend . 0
1

(ii) Dividing this 10 by 2, we get 6 as the second digit of the quotient ; and, as there is no remainder (to be prefixed thereto), we take up that digit 5 itself as our next Dividend. 0 5
1

(iii) So, the next quotient--digit is 2 ; and the remainder is 1. We, therefore, put 2 down as the third digit of the quotient and prefix the remainder (1) to that quotient-digit (2) and thus have 12 as our next Dividend. . 0 5 2
1 1

(iv) This gives us 6 as quotient-digit and zero as Remainder. So, we set 6 down as the fourth digit of the quotient ; and as there is no remainder to be prefixed thereto, we take the 6 itself as our next digit for division. . 0 5 2 6 3 1
1 1 1

(v) That gives us 1 and 1 as Quotient and Remainder respectively. We therefore put 1 down as the 5th quotient-digit, prefix the 1 thereto and have 11 as our next Dividend. 0 5 2 6 3 1 5
1 1 1 1

(vi-xvii) Carrying this process of straight, continuous division by 2, we get 2 as the 17th quotient-digit and 0 as remainder.

(xviii) Dividing this 2 by 2, we get 1 as 18th quotient digit and 0 as remainder. But this is exactly what we began with. This means that the decimal begins to repeat itself from here. So, we stop the mental-division process and put down the usual recurring symbols (dots

0̇ 5 2 6 3 1 5 7 8
1 1 1 1 1 1
9 4 7 3 6 8 4 2 1
1 1 1

on.the 1st and 18th digits) to show that the whole of it is a circulating decimal.

Note that, in the first method (i.e. of multiplication), each surplus digit is *curried* over to the *left* and that, in the second method (i.e. of division), each remainder is *prefixed* to the right (i.e. just immediately to the left of the next dividend digit),

C. A Further short-cut. :

This is not all. As a matter of fact, even this much or rather, this little work (of mental multiplication or division) is not really necessary. This will be self-evident from sheer observation.

Let us put down the first 9 digits of the answer in one horizontal row above and the other 9 digits in another horizontal row just below and observe the fun of it. We notice that each set of digits (in the upper row and the lower row) totals 9. And this means that, when just half the work has been completed (by either of the *Vedic one-line* methods), the other half need *not* be obtained by the same process but is mechanically available to us by subtracting from 9 each of the digits already obtained ! And *this means a lightening of the work still further (by 50%).*

0	5	2	6	3	1	5	7	8
9	4	7	3	6	8	4	2	$\dot{1}$
9	9	9	9	9	9	9	9	9

Yes; but how should one know that the task is exactly half-finished so that one may stop the work (of multiplication or division, as the case may be) and proceed to reel off the remaining half of the answer by subtracting from 9 each of the digits already obtained ? And the answer is—as we shall demonstrate later on—that, in either method, if and as soon as we reach the difference between the numerator and the denominator (i.e. 19—1=18), we shall have completed *exactly half the* work; and, with this knowledge, we know exactly when and where we may stop the multiplication or division work and when and where we can begin reeling off the complements from 9 (as the remaining digits of the answer) !

Thus both in the multiplication method and in the division method, we reach **18** when we have completed **half** the work and can begin the mechanical-subtraction device (for the other half).

Details of these principles and processes and other allied matters, we shall go into, in due course, at the proper **place.** In the meantime, the student will find **it** both interesting and profitable to know this rule and turn it into good account from time to time (as the occasion may demand or justify).

Second Example:

Case 2 1/29 } Let us now take another case of a similar type (say, 1/29) where too the **denominator** ends in **9.**

By the *Current* method :—

$$29)\,1.00(.0\;3\;4\;4\;8\;2\;7\;5\;8\;6\;2\;0\;6\;8\;9\;6\;5\;5\;1\;7\;2\;4\;1\;3\;7\;9\;3\;\dot{1}$$

87

130	180		
116	174		
140	60	150	
116	58	145	
240	200	50	110
232	174	29	87
80	260	210	230
58	232	203	203
220	280	70	270
203	261	58	261
170	190	120	90
145	174	116	87
250	160	40	30
232	145	29	29
180	150	110	1

By the Vedic one-line *Mental* method

A. First Method

$$\frac{1}{29} = .\dot{0}\;3\;4\;4\;8\;2\;7\;5\;8\;6\;2\;0\;6\;8$$
1 1 1 2 2 1 2 1 2 2 2

$$9\;6\;5\;5\;1\;7\;2\;4\;1\;3\;7\;9\;3\;\dot{1}$$
1 1 1 2 1 1 2 2

B. Second Method.

$$\frac{1}{29} = .\dot{0}\;3\;4\;4\;8\;2\;7\;5\;8\;6\;2\;0\;6\;8$$
1 1 1 2 2 1 2 1 2 2 2

$$9\;6\;5\;5\;1\;7\;2\;4\;1\;3\;7\;9\;3\;\dot{1}$$
1 1 2 1 1 2 2

This 'is the whole working (by both the processes). The **procedures** are explained on the next page.

A. Explanation of the First Method:

Here too, the last digit of the denominator is 9; but the penultimate one is **2**; and one more than that means **3.** So, **3** is our *common—i.e. uniform—multiplier* this time. And, following the same procedure as in the case of **1/19,** we put down, **1** as the last (i.e. the right-hand-most) digit of the answer and carry on the multiplication continually (leftward) by **3** ("carrying" the left-hand extra side-digit—if any—over to the left) until the Recurring Decimal actually manifests itself as such. And we find that, by our mental one-line process, waget the same **28** digit-answer as we obtained by **28** steps of cumbrous and tedious working according to the current system, as shown on the left-hand side margin on the previous page.

Our *modus-operandi-chart* herein reads as follows :—

$\frac{1}{29}$ = . 0 3 4 4 8 2 7 5 8 6 2 0 6 8
1 1 1 2 1 2 2 2
9 6 5 5 1 7 2 4 1 3 7 9 3 ;
1 1 1 2 1 1 2 2

B. Explanation of the Second Method:

The Division—process to be adopted here is exactly the same as in the case of **1/19** ; but the Divisor (instead of the multiplier) is uniformly **3** all through. And the chart reads as follows :—

$\frac{1}{29}$ = . $\dot{0}$ 3 4 4 8 2 7 5 8 6 2 0 6 8
1 1 1 2 2 1 2 1 2 2 2
9 6 5 5 1 7 2 4 1 3 7 9 3 .1
1 1 1 2 1 1 2 2

C. The Complements *from* Nine:

Here too, we find that the two halves are all complements of each other (from **9**). So, this fits in too.

∴ $\frac{1}{29}$ = $\dot{0}$ 3 4 4 8 2 7 5 8 6 2 0 6 8
9 6 5 5 1 7 2 4 1 3 7 9 3 $\dot{1}$

9 9 9 9 9 9 9 9 9 9 9 9 9 9

Third *Example* :

Case 3 }
1/49 } By the *current* system.

```
49)1.00( 020408163265306122448
     98  979591836734693877551
     ---
    200
    196
    ---
    400
    392
    ---
     80        120
     49         98
    ---        ---
    310        220
    294        196
    ---        ---
    160        240        410
    147        196        392
    ---        ---        ---
    130        440        180        190
     98        392        147        147
    ---        ---        ---        ---
    320        480        330        430
    294        441        294        392
    ---        ---        ---        ---
    260        390        360        380
    245        343        343        343
    ---        ---        ---        ---
    150        470        170        370
    147        441        147        343
    ---        ---        ---        ---
    300        290        230        270
    294        245        196        245
    ---        ---        ---        ---
     60        450        340        250
     49        441        294        245
    ---        ---        ---        ---
    110         90        460         50
     98         49        441         49
    ---        ---        ---        ---
    120        410        190          1
                                      ---
```

By the Vedic one-line Mental Method :

Our multiplier or divisor (as the case may be) is now **5** (i.e. one more than the penultimate digit). So, **A.** (By multiplication leftward from the right) by **5**, we have—

$\frac{1}{49}$ = . 0 2 0 4 0 8 1 6 3 2 6 5 3 0 6 1 2 2 4 4 8
9 7 9 5 9 1 8 3 6 7 3 4 6 9 3 8 7 7 5 5 1̇
3 4 2 4   4 1 3 3 1 2 3 4 1 4 3 3 2 2

OR B. (By DIVISION rightward from the left) by **6** :—

$\frac{1}{49}$ = . 6 2 0 4 0 8 1 6 3 2 8 5 3 0 6 1 2 2 4 4 8
1   2   4   3 1 1 3 2 1   3   1 1 2 2 4 4
9 7 9 5 9 1 8 3 6 7 3 4 6 9 3 8 7 7 5 6 .

Note:—At this point, in all the **3 processes**, we find that we have reached **48** (the difference between the numerator and the denominator). This means that half the work (of multiplication or division, as the case may be) has been completed and that we may therefore stop that process and may begin the easy and mechanical process of obtaining the remaining digits of the answer (whose total number of digits is thus found to be **21+21=42**). And yet, the remarkable thing is that the current system takes **42** steps of elaborate and cumbrous dividing (with a series of multiplications and subtractions and with the risk of the failure of one or more "trial digits" of the Quotient and so on) while a single, straight and continuous process—of multiplication or division—(by a single multiplier or divisor) is quite enough in the Vedic method.

The complements from nine are also there.

But this is not all. Our readers will doubtless be surprised to learn—but it is an actual fact—that there are, in the Vedic system, still simpler and easier methods by which, without doing even the infinitely easy work explained hereinabove, we can put down digit after digit of the answer, right from the very start to the very end.

But, as these three examples (of $\frac{1}{19}$, $\frac{1}{29}$, and $\frac{1}{49}$) have been dealt with and explained at this stage, not in the contemplated regular sequence but only by way of preliminary *demonstration* for the satisfaction of a certain, natural and understandable, nay, perfectly justifiable type of purely intellectual curiosity, we do not propose to go—here and now—into a further detailed and elaborate, comprehensive and exhaustive exposition of the other astounding processes found adumbrated in the Vedic mathematical Sūtras on this particular subject. We shall hold them over to be dealt with, at their own appropriate place, in due course, in a later chapter.

ॐ तत् सत्

ARITHMETICAL COMPUTATIONS

CHAPTER II

MULTIPLICATION (by '*Nikhilam*' etc. *Sūtra*)

Pass we now on to a systematic exposition of certain salient, interesting, important and necessary formulae of the utmost value and utility in connection with arithmetical calculations etc., beginning with the processes and methods described in the Vedic mathematical Sūtras.

At this point, it will not be out of place for us to repeat that there is a GENERAL formula which is simple and easy and can be applied to all cases; but there are also SPECIAL cases--or rather, types of cases—which are simpler still and which are, therefore, here first dealt with.

We may also draw the attention of all students (and teachers) of mathematics to the well-known and universal fact that, in respect of arithmetical multiplications, the usual present-day procedure everywhere (in schools, colleges and universities) is for the children (in the primary classes) to be told to cram up--or "get by heart"—the multiplication-tables (up to 16 times 16, 20×20 and so on). But, according to the Vedic system, the multiplication tables are not really required above 5×5. And a school-going pupil who knows simple addition and subtraction (of single-digit numbers) and the multiplication-table up to five times five, can *improvise* all the necessary multiplication-tables for himself at any time and can himself do all the requisite multiplications involving bigger multiplicands and multipliers, with the aid of the relevant simple Vedio formulae which enable him to get at the required products, very easily and speedily—nay, practically, immediately. The Sūtras are very short; but, once one understands them and the *modus operandi* inculcated therein for their practical application, the whole thing becomes a sort of children's play and ceases to be a "problem".

1. Let us first take up a very easy and simple illustrative example (i.e. the multiplication of single-digit numbers above 5) and see how this can be done without previous knowledge of the higher multiplications of the multiplication-tables.

The Sūtra reads : निखिलं नवतश्चरमं दशतः (Nikhilaṁ Navataścaramaṁ Daśataḥ) which, literally translated, means; "all from 9 and the last from 10"! We shall give a detailed explanation, presently, of the meaning and applications of this cryptical-sounding formula. But just now, we state and explain the actual procedure, step by step

Suppose we have to multiply 9 by 7.

(i) We should take, as Base for our calculations, that power of 10 which is nearest to the numbers to be multiplied. In this 10 itself is that power ;

(10)
9—1
7—3
6/3

(ii) Put the two numbers 9 and 7 above and below on the lefthand side (as shown in the working alongside here on the right-hand side margin) ;

(iii) Subtract each of them from the base (10) and write down the remainders (1 and 3) on the right-hand side with a connecting ***minus*** sign (—) between them, to show that the numbers to be multiplied are both of them less than 10.

(iv) The product will have two parts, one on the left side and one on the right. A vertical dividing line may be drawn for the purpose of demarcation of the two parts.

(v) Now, the left-hand-side digit (of the answer) can be arrived at in one of 4 ways:—

(a) Subtract the base 10 from the sum of the given numbers (9 and 7 i.e. 16). And put (16—10) i.e. 6, as the left-hand part of the answer ;

9+7—10=6

OR (b) Subtract the sum of the two deficiencies (1+3=4) from the base (10). You get the same answer (6) again ;

10—1—3=6

OR (c) Cross-subtract deficiency (3) on the second row from the original number (9) in the first row. And you find that you have got (9—3) i.e. 6 again. **9—3=6**

OR (d) Cross-subtract in the converse way (i.e. 1 from 7). And you get 6 again as the left-hand side portion of the required answer. **7—1=6**

Note:—This availablity of the same result in several easy ways is a very common feature of the Vedic system and is of great advantage and help to the student (as it enables him to test and verify the correctness of his answer, step by step).

(vi) Now, vertically multiply the two deficit figures (1 and 3). The product is 3. And this is the right-hand-side portion of the answer.

(vii) Thus **9×7=63.**

(10) 9—1
7—3
6/3

This method holds good in all cases and is, therefore, capable of infinite application. In fact, old historical traditions describe this cross-subtraction process as having been responsible for the acceptance of the × mark as the sign of multiplication.

(10)
9—1
×
7—3
6 / 3

As further illustrations of the same rule, note the following examples:—

9—1	9—1	9—1	9—1	8—2	8—2	8—2	7—3
9—1	8—2	6—4	5—5	8—2	7—3	6—4	7—3
8/1	7/2	5/4	4/5	6/4	5/6	4/8	4/9

This proves the correctness of the formula. The algebraical explanation for this is very simple :—

$$(x-a)(x-b) = x(x-a-b)+ab.$$

A slight difference, however, is noticeable when the vertical multiplication of the deficit digits (for obtaining the right-hand-side portion of the answer) yields a product consisting of more than one digit. For example, if and when we have to multiply 6 by 7, and write it down as usual :—

$$\begin{array}{l} 7-3 \\ 6-4 \\ \hline 3/_{1}2 \end{array}$$

we notice that the required vertical multiplication (of 3 and 4) gives us the product 12 (which consists of 2 digits ; but, as our base is 10 and the right-hand-most digit is obviously of units, we are entitled only to one digit (on the right-hand side).

This difficulty, however, is easily surmounted with the usual multiplicational rule that the surplus portion on the left should always be "carried" over to the left. Therefore, in the present case, we keep the 2 of the 12 on the right hand side and "carry" the 1 over to the left and change the 3 into 4. We thus obtain 42 as the actual product of 7 and 6.

$$\begin{array}{l} 7-3 \\ 6-4 \\ \hline 3/_{1}2 = 4/2 \\ \hline \end{array}$$

A similar procedure will naturally be required in respect of other similar multiplications :—

$$\begin{array}{llll} 8-2 & 7-3 & 6-4 & 6-4 \\ 5-5 & 5-5 & 6-4 & 5-5 \\ \hline 3/_{1}0 = 4/0 & 2/_{1}5 = 3/5 & 2/_{1}6 = 3/6 & 1/_{2}0 = 3/0 \\ \hline \end{array}$$

This rule of multiplication (by means of cross-subtraction for the left-hand portion and of vertical multiplication for the right-hand half), being an actual application of the absolute algebraic identity :—$(x+a)(x+b) = x(x+a+b)+ab$, can be

extended further without any limitation. Thus, as regards numbers of two digits each, we may notice the following specimen examples :—

N.B. The base now required is 100.

91—9	93—7	93—7	93—7	89—11	91—9	93—7
91—9	92—8	93—7	94—6	95— 5	96—4	97—3
82/81	85/56	86/49	87/42	84/66	87/36	90/21

92—8	88—12	78—22	88—12	56—44	67—33	25—75
98—2	98— 2	97— 3	96— 4	98— 2	97— 3	99— 1
90/16	86/24	75/66	84/48	64/88	64/99	24/75

Note 1:—In all these cases, note that both the cross-subtractions always give the same remainder (for the left-hand-side portion of the-answer).

Note 2 :—Here too, note that the vertical multiplication (for the right-hand side portion of the product) may, in some cases, yield a more-than-two-digit product ; but, with 100 as our base, we can have only two digits on the right-hand side. We should therefore adopt the same method as before (i.e. keep the permissible two digits on the right-hand side and "carry" the surplus or extra digit over to the left) (as in the case of ordinary addition, compound addition etc.) Thus—

88—12	88—12	25—75
88—12	91— 9	98— 2
$76/_{1}44 = 77/44$	$79/_{1}08 = 80/08$	$23/_{1}50 = 24/50$

Note:—Also, how the meaning of the Sūtra comes out in all the examples just above dealt with and tells us how to write down immediately the deficit figures on the right-hand side. The rule is that all the other digits (of the given original numbers) are to be subtracted from 9 but the last (i.e. the right hand-most one)

should be deducted from 10. Thus, if **63** be the given number, the deficit (from the base)' is **37**; and so on. This process helps us in the work **of** ready on-sight subtraction **and** enables us to put the deficiency down immediately.

A new point has **now to** be taken **into** consideration **i.e.** that, just as the process of **vertical multiplication** may yield a ***larger*** number of digits in the product than is permissible (and this contingency has been provided for), so, it may similarly yield a product consisting of a ***smaller*** number of digits than we are entitled to. What is the remedy herefore ? Well, this contingency too has been provided for. **And** the remedy is—as in the case of decimal multiplications—merely the filling up of all such vacancies with Zeroes. Thus,

99 – 1	**98 – 2**	**96—4**	**97—3**
97 – 3	**99 – 1**	**98 – 2**	**97—3**
96/03	**97/02**	**94/08**	**94/ 09**

With these **3** procedures (for meeting the **3** possible contingencies in question i.e. of normal, abnormal and sub-normal number of digits in the vertical-multiplication-products) and with the aid of the subtraction-rule (i.e. of all the digits from 9 and the last one from **10**, for writing down the amount of the deficiency from the base), we can extend this multiplication-rule to numbers consisting of a larger number of digits, thus—

888—112	**879—121**	**697 – 303**	**598—402**
998 – 002	**999 – 001**	**997 – 003**	**998—002**
886/224	**878/121**	**694/909**	**596/804**

988—012	**888—112**		**112—888**
988—012	**991—009**		**998— 2**
976/144	**879/$_1$008**	**=880/008**	**110/$_1$776=111/776**

988—012	998—002	9997—0003
998—002	997—003	9997—0003
986/024	995/006	9994/0009

99979—00021	999999997—000000003
99999—00001	999999997—000000003
99978/00021	999999994/000000009

Yes; but, in all these cases, the multiplicand and the multiplier are just a little below a certain power of ten (taken as the base). What about numbers which are above it ?

And the answer is that the same procedure will hold good there too, except that, instead of ***cross-subtracting,*** we shall have to ***cross-add.*** And all the other rules (regarding digit-surplus, digit-deficit etc.,) will be exactly the same as before. Thus,

12+2	13+3	11+1	16+6	18+8	108+8	111+11
11+1	12+2	15+5	11+1	11+1	108+8	109+9
13/2	15/6	16/5	17/6	19/8	116/64	120/99

16+6	17+7	18+8
12+2	12+2	12+2
$18/{}_{1}2$ =19/2	$19/{}_{1}4$ =20/4	$20/{}_{1}6$ =21/6

18+8	19+9	1005+5	1016+16
18+8	19+9	1009+9	1006+6
$26/{}_{6}4$=32/4	$28/{}_{8}1$=36/1	1014/045	1022/096

In passing, the algebraical principle involved may be explained as follows:--

(~ + a)(x+b)=x(x+a+b)+ab.

Yes; but if one of the numbers is *above* and the other is ***below*** a power of 10 (the base taken), what then ?

The answer is that the plus and the minus will. on multiplication, behave as they always do and produce a nunus-product and that the right-hand portion (obtained by vertical multi-

plication) will therefore have to be subtracted. A vinculum may be used for making this clear. Thus,

12+2	108+8	107+7	102+2
8—2	97—3	93—7	98—2
10/$\overline{4}$=96	105/$\overline{24}$=104/76	100/$\overline{49}$=99/51	100/$\overline{04}$=99/96

1026+26	1033+33
997— 3	977— 3
1023/$\overline{078}$=1022/922	1030/$\overline{099}$=1029/901

10006+6
9999—1

10005/00006=9994/99994

Note:—Note that even the subtraction of the vinculum-portion may be easily done with the aid of the Sūtra under discussion (i.e. all from 9 and the last from 10).

Multiples **and** *sub-multiples* :

Yes ; but, in all these cases, we find both the multiplicand and the multiplier, or at least one of them, very near the base taken (in each case) ; and this gives us a small multiplier and thus renders the multiplication very easy. What about the multiplication of two numbers, neither of which is near a convenient base ?

The needed solution for this purpose is furnished by a small 'Upasūtra' (or sub-formula) which is so-called because of its practically axiomatic character.

This sub-shtra consists of only one word आनुरूप्येण (Ānurūpyeṇa) which simply means "Proportionately". In actual application, it connotes that, in all cases where there is a rational ratio-wise relationship, the ratio should be taken into account and should lead to a proportionate multiplication or division as the case may be.

In other words, when neither the multiplicand nor the multiplier is sufficiently near a convenient power of 10 (which can suitably serve us as a base), we can take a convenient mul-

tiple or sub-multiple of a suitable base, as our "Working Base", perform the necessary operation with its aid and then multiply or divide the result proportionately (i.e. in the same proportion as the original base may bear to the working base actually used by us). A concrete illustration will make the *modus operandi* clear.

Suppose we have to multiply 41 by 41. Both these numbers are so far away from the base 100 that by our adopting that as our actual base, we shall get 59 and 59 as the deficiency from the base. And thus the consequent vertical multiplication of 59 by 59 would prove too cumbrous a process to be permissible under the Vedic system and will be positively inadmissible.

We therefore, accept 100 merely as a theoretical base and take sub-multiple 50 (which is conveniently near 41 and 41) as our working basis, work the sum up accordingly and then do the proportionate multiplication or division, for getting the correct answer.

Our chart will then take this shape :—

$$\frac{100}{2} = 50$$

$$\begin{array}{r} 41-9 \\ 41-9 \\ \hline 2)\,32/81 \\ \hline 16/81 \\ \hline \end{array}$$

(i) We take 50 as our working base.

(ii) By cross — subtraction, we get 32 on the left-hand side.

(iii) As 50 is a half of 100, we therefore divide 32 by 2 and put 16 down as the real left-hand-side portion of the answer.

(iv) The right-hand-side portion (81) remains un-affected.

(v) The answer therefore is 1681.

OR, *secondly,* instead of taking 100 as our theoretical base and its half (50)

as our working base (and dividing 32 by 2), we may take **10**, as our theoretical base and its multiple **50** as our working base and ultimately multiply **32** by **5** and get **160** for the left-hand side. And as **10** was our theoretical base and we are therefore entitled to only one digit on the right hand side, we retain **1** (of the **81**) on the right hand side, "carry" the **8** (of the **81**) over to the left, add it to the **160** already there and thus obtain **168** as our left-hand-side portion of the answer. The product of **41** and **41** is thus found to be **1681** (the same as we got by the first method).

$$\begin{array}{l} 10\times5=50 \\ \hline 41-9 \\ 41-9 \\ \hline 32\,/\,1 \\ \times\ 5/8 \\ \hline 160/_{8}1=1681 \\ \hline \end{array}$$

OR, thirdly, instead of taking **100** or **10** as our theoretical base and **50** (a sub-multiple or multiple thereof) as our working base, we may take **10** and **40** as the bases respectively and work at the multiplication as shown (on the margin) here. And we find that the product is **1681** (the same as we obtained by the first and the second methods).

$$\begin{array}{l} 10\times4=40 \\ 41+1 \\ 41+1 \\ \hline 42\,/\,1 \\ \times\ 4/ \\ \hline 168/1 \\ \hline \end{array}$$

Thus, as we get the same answer (**1681**) by all the three methods; we have option to decide—according to our own convenience—what theoretical base and what working base we shall select for ourselves.

As regards the principle underlying and the reason behind the vertical-multiplication operation (on the right-hand-side) remaining unaffected and not having to be multiplied or divided "proportionately" a very simple' illustration will suffice to make this clear.

Suppose we have to divide **66** successively by **2, 4, 8, 16, 32** and **64** (which bear **a** certain internal **ratio** or ratios among themselves). We may write down our table of answers as follows :—

$$\frac{65}{2}=32\frac{1}{2};\ \frac{65}{4}=16\frac{1}{4};\ \frac{65}{8}=8\frac{1}{8};\ \frac{65}{16}=4\frac{1}{16};$$

and

$$\frac{65}{32}=2\frac{1}{32};\ \text{and}\ \frac{66}{64}=1\frac{1}{64}\ \text{R}\ \textit{is constant.}$$

We **notice** that, as the denominator (**i.e.** the divisor) goes on ***increasing*** in a certain ratio, the quotient goes on *decreasing*, proportionately ; but the ***remainder remains constant.*** And this is why it is rightly called the ***reminder*** (**शिष्यते शेषसंज्ञः ॥**).

The following additional examples will serve to illustrate the principle and process of **आनुरूप्येण** (**i.e.** the selecting of a multiple or sub-multiple as our working base and doing the multiplication work in this way).

(*1*) 49×49

Working ***Base*** 100/2=50

49−1
49−1
2)48/01
24/01

(2) OR 49 X 49

Working Base *10×5=50*

49−1
49−1
48 / I
× 5 /
240 / 1

(*3*) 46×46

Working Base *100/2=50*

46−4
46−4
2)42/16
21/16

(4) OR 46×46

Working ***Base*** *10×5=50*

46−4
46−4
42 / $_1$6
× E /
210 / 16=211/6

(6) 46X44

Working Base 10X5=50

46—4
44—6

40 / $_{2}4$
×5 /

200 / $_{2}4$
= 202 / 4

(6) *OR* 46x44

Working Base 100/2=50

46—4
44—6

2) 40 / 24

20 / 24

(7) 59x59

Working Base 10×6=60

59—1
59—1

58 / 1
× 6

348 / 1

(8) *OR* 59X59

Working Base 10X5=50

59+9
59+9

68 / 1
×5 / 8

348 / 1

(9) *OR* 59X59

Working Base 100/2=50

59+9
59+9

2) 68 / 81

34 / 81

(10) 23×23

Working Base 10×2=20

23+3
23+3

26 / 9
X2

52 / 9

(11) 54×46

Working Base 10×5=50

54+4
46—4

50 / $\overline{16}$
×5

250 / $\overline{16}$

=24 / 84

(12) *OR* 54x46

Working Base 100/2=50

54+4
46—4

2) 50 / $\overline{16}$

25 / $\overline{16}$

= 24 / 84

(13) 19×19

Working Base 10×2=20

19−1
19−1

18 / 1
×2 /

=36 / 1

(14) *OR* 19x19

Working Base 10×1

19+9
19+9

28 / $_{8}1$
+8 /

=36 / 1

(15) 62×48

Working Base 10×4=40

62+22
48+ 8

70/$_{17}6$
×4/

280/$_{17}6$

=297/ 6

(16) *OR* 62×48

Working Base 10×6=60

62+ 2
48 −12

50/−24
×6

300/−24
=29/ 76

(17) *OR* 62x48

Working Base 10×5=50

62+12
48− 2

60/−24
X5

300/−24
=29/ 76

(18) *OR* 62×48

Working Base 100/2=50

62+12
48− 2

2) 60/—24

30/−24
= 29/ 76

(19) 23x21

Working Base 10×3=30

23−7
21−9

14/63
X3

42 /$_{6}3$
=48 / 3

(20) *OR* 23×21

Working Baee 10×2=20

23+3
21+1

24 / 3
×2 /

=48 / 3

(21) 249×245

Working Base 1000/4=250

249—1

245—5

4) 244 / 005

=61 / 005

(22) 48×49

Working Base 10×5=50

48—2

49—1

47 / 2

×5

=235 / 2

(23) OR 48×49

Working Base 100/2=50

48—2

49—1

2) 47 / 02

$23\frac{1}{2}$ / 02 = 23/52

Note:—Here 47 being odd, its division by 2 gives us a fractional quotient $23\frac{1}{2}$ and that, just as half a rupee or half a pound or half a dollar is taken over to the right-hand-side (as 8 annas or 10 shillings or 60 cents), so the half here (in the $23\frac{1}{2}$) is taken over to the right-hand-side (as 60). So, the answer is 23/62.

(24) 249×246

Working Base 1000/4=250

249—1

246—4

4)245 / 004

$61\frac{1}{4}$ / 004

=61/254

(25) 229×230

Working Base 1000/4=250

229—21

230—20

4)209 / 420

$52\frac{1}{4}$ / 420

=52/670

Note:—In the above two cases, the $\frac{1}{4}$ on the left hand side is carried over to the right hand (as 250).

The following additional (worked out) examples will serve to further elucidate the principle and process of multiplication according to the Vedic Siitra (*'Nikhilaṁ'* etc) and facilitate the student's practice and application thereof :—

(1) 87965 × 99998

87965—12035
99998— 2

=87963 / 24070

(2) 48 × 97

48— 52
97— 3

45 / $_{1}$56
=46/ 56

(3) 72 × 95

72 — 28
95 — 5

67 / $_{1}$40
=68/40

(4) 889 × 9998

0889—9111
9998— 2

887 /$_{1}$8222

=888 / 8222

(5) 77 × 9988

0077— 9923
9988— 0012

65 /$_{11}$9076

=76 / 9076

(6) 299 × 299

W. B. 100 × 3=300

299— 1
299— 1

298 / 01
× 3 /

=894 / 01

(7) 687 × 699

W. B. 100 × 7=700

687—13
699— 1

686 / 13
× 7 /

=4802 / 13

(8) 128 × 672

W. B. 100 × 7=700

128— 672
672— 28

100/ 16
× 7 / 160

700 /$_{160}$16

=860 / 16

(*9*) 231×582
W. B. 100×6=600
231− 369
582− 18

213 / $_{66}$42
×6 /

1278 / $_{66}$42

=1344 / 42

(*10*) 362×785
W. B. 100×8=800
362−438
785− 15

347 / $_{65}$70
×8 /

2776 / $_{65}$70

=2841 / 70

(*11*) 532×528
W. B. 100×5=500
532+ 32
528+ 28

560 / $_{8}$96
×5 /

2800 / $_{8}$96

=2808 / 96

(*12*) *OR* 532×528
W. B. 1000/2=500
532+ 32
528+ 28

2) 560 / 896

=280 / 896

(*13*) 532×472
W. B. 100×5=500
532+ 32
472− 28

504/−896
×5/

2520/−896

=251 / 104

(*14*) *OR* 532×472
W. B. 1000/2=500
532+ 32
472− 28

2)504/−896

252/−896
=251 / 104

(15) 235×247
W. B. 1000/4=250
235− 16
247− 3

4) 232 / 045

=58 / 045

(16) 3998×4998
W. B. 10000/2=5000
3998−1002
4998− 2

2) 3996 / 2004

=1998 / 2004

(*17*) 19×499

W. B. 100×5=500

```
 19—481
499—  1
--------
 18 / ₄81
 ×5
--------
 =9 / 481
```

(*18*) *OR* 19×499

W. B. 1000/2=500

```
   19—481
  499—  1
-----------
2) 18 / 481
-----------
   =9 / 481
```

(*19*) 635×502

W. B. 1000/2=500

```
   635+135
   502+  2
--------------
2) 637 / 270
--------------
  318½ / 270
--------------
 =318 / 770
```

(*20*) 18×45

W. B. 100/2=50

```
   18— 32
   45—  5
------------
2) 13 / ₁60
------------
   6½ / ₁60
------------
   =8 /  10
```

(*21*) 389×516

W. B. 1000/2=500

```
  389— 111
  516+  16
--------------
2)405/—1776
--------------
 202½/—1776
--------------
 202—1276
=200 /  724
```

Note :—Most of these examples are quite easy, in fact much easier-by the ऊर्ध्वतिर्यग्भ्याम् (*Ūrdhva-Tiryagbhyām*) *Sūtra* which is to be expounded in the next chapter. They have been included here, merely for demonstrating that they too can be solved by the '*Nikhilaṁ*' *Sūtra* expounded in this chapter.

But before we actually take up the formula and explain its *modus operandi* for multiplication, we shall just now explain a few corollaries which arise out of the '*Nikhilaṁ*' Sūtra which is the subject-matter of this chapter.

The First Corollary :

The first corollary naturally arising out of the '*Nikhilaṁ*' Sūtra reads as follows :—यावदून तावदूनीकृत्य वर्गं च योजयेत् ॥ which means:—"whatever the extent of its deficiency, lessen it still further to that very extent ; and also set up the square (of that deficiency)".

This evidently deals with tho squaring of numbers. A few elementary examples will suffice to make its meaning and application clear :—

Suppose we have to find the square of 9. The following will be the successive stages in our (mental) working :—

(i) We should take up the nearest power of 10 (i. e. 10 itself) as our base.

(ii) As 9 is 1 less than 10, we should decrease it still further by 1 and set it (the 8) down as our left-side portion of the answer. 8/

(iii) And, on the right hand, we put down the square of that deficiency (1^2) 8/1

(iv) Thus $9^2=81$

9—1
9—1
8 / 1

Now, let us take up the case of 8^2. As 8 is 2 less than 10, we lessen it still further by 2 and get 8—2 (i. e. 6) for the left-hand and putting 2^2 (=4) on the right-hand side, we say $8^2=64$

6/
6/4

8—2
8—2
6 / 4

In exactly the same manner, we *say*

$7^2=(7-3)$ / $3^2=4/9$

$6^2=(6-4)$ and $4^2=2/_{1}6=3/6$

5^2—(5—5) and 5^2—$0/_{2}5=25$; and so on

7—3
7—3
4 / 9

Yes; but what about numbers *above* 10? We work exactly as before; but, instead of *reducing* still further by the deficit, we *increase* the number still further by the *surplus* and say :—

$11^2=(11+1)/1^2=12/1$

$12^2=(12+2)/2^2=14/4$

$13^2=(13+3)/3^2=16/9$

$14^2=(14+4)/4^2=18/_16=19/6$

$15^2=(15+5)/5^2=20/_25=225$

$19^2=(19+9)/9^2=28/_81=361$; and so on.

$$\begin{array}{r} 11+1 \\ 11+1 \\ \hline 12/1 \end{array} \qquad \begin{array}{r} 19+9 \\ 19+9 \\ \hline 28/_81=361 \end{array}$$

And then, extending the same rule to numbers of two or more digits, we proceed further and say :—

$91^2=82/81$; $92^2=84/64$; $93^2=86/49$;
$94^2=88/36$; $95^2=90/25$; $96^2=92/16$;
$97^2=94/09$; $98^2=96/04$; $99^2=98/01$;
$108^2=116/64$; $103^2=106/09$;
$989^2=978/121$; $988^2=976/144$; $993^2=986/049$;
$89^2=78/_121=79/21$; $88^2=76/_144=77/44$;
$9989^2=9978/0121$; $9984^2=9968/0256$; $9993^2=$
$=9986/0049$;

The Algebraical Explanation for this is as follows :—

$(a+b)^2=a^2\pm2ab+b^2$

$\therefore 97^2=(100-3)^2=10000-600+9=94/09$;

$92^2=(100-8)^2=10000-1600+64=84/64$;

$108^2=(100+8)^2=10000+1600+64=116/64$; and so on

A Second Algebraical Explanation is as follows :—

$a^2-b^2=(a+b)(a-b)$

$\therefore a^2=(a+b)(a-b)+b^2$

So, if we have to obtain the square of any number (a), we can add any number (b) to it, subtract the same number (b) from it and multiply the two and finally add the square of that number (b) (on the right hand side). Thus, if 97 has

to be squared, we should select such a number (b) as will, by addition or by subtraction, give us a number ending in a zero (or zeros) and thereby lighten the multi-multiplication work. In the present case, if our (b) be 3, a+b will become 100 and a—b will become 94. Their product is 9400; and $b^2=9$ $\therefore 97^2=94/09$. This proves the Corollary.

Similarly, $92^2=(92+8)(92-8)+64=84/64$;
$93^2=(93+7)(93-7)+49=86/49$;
$988^2=(988+12)(988-12)+144=976/144$;
$108^2=(108+8)(108-8)+64=116/64$; and so on.

The Third Algebraical Explanation is based on the *Nikhilaṁ Sūtra* and has been indicated already.

91— 9
91— 9
82 / 81

The following additional sample-examples will further serve to enlighten the student (on this Corollary):—

(1) 19^2
19+9
19+9
28/81
=36 / 1

OR (2) 19^2
19—1
19—1
18 / 1
X 2
=36 / 1

(3) 29^2
29+9
29+9
38 / $_8$1
X 2
=84 / 1

OR (4) 29^2
29—1
29—1
28 / 1
×3
=84 / 1

(5) 49^2
49— 1
49— 1
48 / 1
×5/
=240 / 1

OR (6) 49^2
49— 1
49— 1
2) 48 / 01
=24 / 01

(7) 59^2

$59+9$
$59+9$

$68 / {}_{8}1$
$\times 5$

$340 / {}_{8}1 = 34/81$

OR (8) 59^2

$59+\ 9$
$59+\ 9$

$2)\ 68 / 81$

$=34 / 81$

(9) 41^2

$41+1$
$41+1$

$42 / 1$
$\times 4$

$=168 / 1$

OR (10) 41^2

$41-\ 9$
$41-\ 9$

$2)\ 32 / 81$

$=16 / 81$

(11) 989^2

$989-\ 11$
$989-\ 11$

$=978 / 121$

(12) 775^2

W. B. $100 \times 8 = 800$

$775-\ 25$
$775-\ 25$

$750 / {}_{6}25$
$\times 8$

$=6006 /\ 25$

Note:—All the cases dealt with hereinabove are doubtless of numbers just a little below or just a little above a power of ten or of a multiple or sub-multiple thereof. This corollary is specially suited for the squaring of such numbers. Seemingly more complex and "difficult" cases will be taken up in the next chapter (relating to the *Ūrdhva-Tiryak Sūtra*); and still most "difficult" will be explained in a still later chapter (dealing with the squaring, cubing etc., of bigger numbers).

The Second Corollary.

The second corollary is applicable only to a special case under the first corollary (i. e. the squaring of numbers ending in 6 and other cognate numbers). Its wording is exactly the same as that of the Sūtra which we used at the outset for the conversion of vulgar fractions into their recurring decimal equivalents (i. e. एकाधिकेन पूर्वेण). The Siitra now takes a totally different meaning altogether and, in fact, relates to a wholly different set-up and context altogether.

Its literal meaning is the same as before (i. e. "by one more than the previous one"); but it now relates to the squaring of numbers ending in 5 (e. g. say, 15). Here, the last digit is 5; and the "previous" one is 1. So, one more than that is 2. Now, the Sfitra in this context tells us to multiply the previous digit (1) by one more than itself (i. by 2), So the left-hand side digit is 1×2; and the right-hand side is the vertical-multiplication-product (i.e. 25) as usual.

$$\frac{1/\ 5}{2/25}$$

'thus $15^2=1\times2/25=2/25$.

Similarly, $25^2=2\times3/25=6/25$;
$35^2=3\times4/25=12/25$;
$45^2=4\times5/25=20/25$;
$55^2=5\times6/25=30/25$;
$65^2=6\times7/25=42/25$;
$75^2=7\times8/25=56/25$;
$85^2=8\times9/25=72/25$;
$95^2=9\times10/25=90/25$;
$105^2=10\times11/25=110/25$;
$115^2=11\times12/25=132/25$;
$125^2=156/25$; $135^2=182/25$; $145^2=210/25$;
$155^2=240/25$; $165^2=272/25$; $175^2=306/25$;
$185^2=342/25$; $195^2=380/25$; and so on.

The *Algebraical Explanation* is quite simple and follows straight-away from the *Nikhilaṁ Sūtra* and still more so from the *Ūrdhva-Tiryak* formula to be explained in the next chapter (q.v.).

A sub-corollary to *this Corollary* (relating to the squaring of numbers ending in 5) reads : अन्त्ययोर्दशकेऽपि (*Antyayor-Daśake'pi*) and tells us that the above rule is applicable riot only to the squaring of a number ending in 5 but also to the multiplication of two numbers whose last digits together total 10 and whose previous part is exactly the same.

For example, if the numbers to be multiplied are not 25 and 25, but, say 27 and 23 [whose last digits i.e. 7 & 3 together total 10 and whose previous part is the same namely 2], even then the same rule will apply (i. e. that the 2 should be multiplied by 3 the next higher number. Thus we have 6 as our left-hand part of the answer; and the right-hand one is, by vertical multiplication (as usual)

$7 \times 3 = 21$. And so $27 \times 23 = 6/21$.

$$\begin{array}{r} 27 \\ 23 \\ \hline =6/21 \end{array}$$

We can proceed further on the same lines and say :—

$96 \times 94 = 90/24$; $97 \times 93 = 90/21$; $98 \times 92 = 90/16$;
$99 \times 91 = 90/09$; $37 \times 33 = 12/21$; $79 \times 71 = 56/09$;
$87 \times 83 = 72/21$; $114 \times 116 = 132/24$; and so on

This sub-corollary too is based on the same *Nikhilaṁ* Sfitra ; and harder examples thereof will more appropriately come under the *Ūrdhva-Tiryak* formula of the next chapter (or the still later chapter on more difficult squarings and cubings).

At this point, however, it may just be pointed out that the above rule is capable of further application and come in handy, for the multiplication of numbers whose last digits (in sets of 2,3 and so on) together total 100, 1000 etc. For example—

$$\left.\begin{array}{r} 191 \times 109 = 20/819 \\ 793 \times 707 = 560/651 \\ 884 \times 816 = 720/_{1}344 = 721/344. \end{array}\right\}$$

N. B.—Note the added zero at the end of the left-hand-side of the answer.

The Third Corollary :

Then comes a Third Corollary to the *Nikhilaṁ Sūtra*, which relates to a very special type of multiplication and which is not frequently in requisition elsewhere but is often required in mathematical astronomy etc. The wording of the sub-sūtra (corollary) एकन्यूनेन पूर्वेण (*Ekanyūnena Pūrveṇa*) sounds as

if it were the converse of the ***Ekādhika Sūtra*** . It actually is ; and it relates to and provides for multiplications wherein the multiplier-digits consist entirely of nines. It comes up under three different headings as follows :—

***The First* case .**

The annexed table of products produced by the single-digit multiplier **9** gives us the necessary clue to an under-standing of the **Sūtra** :—

9× 2=1	8
9× 3=2	
9× 4=3	
9× 5=4	5
9× 6=5	4 3
9× 7=6	
9× 8=7	2
9× 9=8	1
9×10=9	0

We observe that the left-hand-side is invariably one less than the multiplicand and that the right-sidedigit is merely the complement of the left-handside digit from **9**. And this tells us what to do to get both the portions of the product.

The word '*Pūrṇa*' in this context has another technico-terminological usage and simply means the "multiplicand" (while the word 'Apara' signifies the multiplier).

The meaning of the sub-corollary thus fits in smoothly into its context i. e. that the multiplicand has to be decreased by **1** ; and as for the right-hand side, that is mechanically available by the subtraction of the left-hand-side from *9* (which is practically a direct application of the ***Nikhilaṁ Sūtra***).

As regards multiplicands and multipliers of **2** digits each, we have the following table of products :—

11×99=10	*89*	=(11−1)/99−(11−1)=1089
12×99=11	88	
13×99=12	*87*	
14×99=13	*86*	
15×99=14	85	
16×99=15	*84*	
17×99=16	*83*	
18×99=17	*82*	
19×99=18	*81*	
20×99=19	*80*	

And this table shows that the rule holds good here too. And by similar continued observation, we find that it is uniformly applicable to all cases, where the multiplicand and the multiplier consist of the same number of digits. In fact, it is a simple application of the *Nikhilaṁ Sūtra* and is bound to apply.

7– 3	*77– 23*	*979– 021*
9– 1	*99– 1*	*999– 1*
6 / 3	*76 / 23*	*978 / 021*

We are thus enabled to apply the rule to all such cases and say, for example :—

777	9879	*1203579*
999	9999	*9999999*
776/223	9878/0121	1203578/8796421

9765431	1 2 3 4 5 6 7 8 0 9
9999999	9 9 9 9 9 9 9 9 9 9
9765430/0234569	1234567808/8765432191

Such multiplications (involving multipliers of this special type) come up in advanced astronomy etc ; and this sub-formula (*Ekanyūnena Pūrveṇa*) is of immense utility therein.

The Second Case :

The second case falling under this category is one wherein the multiplicand consists of a smaller number of digits than the multiplier. This, however, is easy enough to handle ; and all that is necessary is to fill the blank (on the left) in with the required number of zeroes and proceed exactly as before and then leave the zeroes out. Thus—

7	79	798	*79*
99	999	99 999	9999999
06/93	078/921	00797/99202	0000078/9999921

The Third Case:

(To be omitted during a first reading).

The third case coming under this heading is one where the multiplier contains a smaller number of digits than the multiplicand. Careful observation and study of the relevant table of products gives us the necessary clue and helps us to understand the correct application of the Sūtra to this kind of examples.

Column 1		Column 2		Column 3
11×9= 9	9	21×9=18	9	37×9=33/3
12×9=10	8	22×9=19	8	46×9=41/4
13×9=11	7	23×9=20	7	55×9=49/5
14×9=12	6	24×9=21	6	64×9=57/6
15×9=13	5	25×9=22	5	73×9=65/7
16×9=14	4	26×9=23	4	82×9=73/8
17×9=15	3	27×9=24	3	91×9=81/9
18×9=16	2	28×9=25	2	and so on
19×9=17	1	29×9=26	1	
20×9=18	0	30×9=27	0	

We note here that, in the first column of products where the multiplicand starts with 1 as its first digit the left-hand-side part (of the product) is uniformly 2 less than the multiplicand; that, in the second column (where the multiplicand begins with 2,) the left-hand side part of the product is exactly 3 less; and that, in the third column (of miscellaneous first-digits) the difference between the multiplicand and the left-hand portion of the product is invariable one more than the excess portion to the extreme left of the dividend.

The procedure applicable in this case is therefore evidently as follows:—

(i) Divide the multiplicand off by a vertical line—into a right-hand portion consisting of as many digits as the multiplier; and subtract from the multiplicand one more than the whole excess portion (on the left). This gives us the left-hand-side portion of the product.

OR take the *Ekanyūna* and subtract therefrom the previous (i. e. the excess) portion on the left; and

(ii) Subtract the **right-hand-side** part of the multiplicand by the *Nikhilaṁ* rule. This will give you the right-hand-side of the product.

The following **examples** will make the **process** clear :—

(*1*) 43×9

4 : 3 :
: – 5 : 3

3 : 8 : 7

(2) 63×9

6 : 3 :
: –7 : 3

5 : 6 : 7

(3) 122×9

12 : 2 :
–1 : 3 : 2

10 : 9 : 8

(*4*) 112×99

1 : 12 :
– : 2 : 12

1 : 10 : 88

(*5*) 11119×99

111 : 19 :
–1 : 12 : 19

110 : 07 : 81

(*6*) 4599×99

45 : 99 :
: –46 : 99

45 : 53 : 01

(*7*) 15639×99

156 : 39 :
–1 : 57 : 39

154 : 82 : 61

(*8*) 25999×999

25 : 999 :
: –26 : 999

24 : 973 : 001

(*9*) 777999×9999

77 : 7999 :
: –78 : 7999

77 : 7921 : 2001

(*10*) 111011×99

1110 : 11 :
–11 : 11 : 11

1099 : 00 : 89

(*11*) 1000001×99999

10 : 00001 :
: –11 : 00001

9 : 99990 : 99999

CHAPTER III

MULTIPLICATION

(by Ūrdhva-Tiryak Sūtra)

Having dealt in fairly sufficient detail with the application of the *Nikhilaṁ Sūtra* etc., to special cases of multiplication, we now proceed to deal with the ऊर्ध्वतिर्यग्भ्याम् (*Ūrdhva Tiryagbhyām*) *Sūtra* which is the General Formula applicable to all cases of multiplication (and will also be found very useful, later on, in the division of a large number by another large number).

The formula itself is very short and terse, consisting of only one compound word and means "vertically and cross-wise". The applications of this brief and terse Sūtra are manifold (as will be seen again and again, later on). First we take it up in its most elementary application (namely, to Multiplication in general).

A simple example will suffice to clarify the *modus operandi* thereof. Suppose we have to multiply **12** by **13.**

(i) We multiply the left-hand-most digit (1) of the multiplicand *vertically* by the left-hand-most digit (1) of the multiplier, get their product (1) and set it down as the left-hand-most part of the answer.

```
12
13
——————
1 : 3+2 : 6=156
——————
```

(ii) We then multiply **1** and **3**, and **1** and **2** *cross-wise*, add the two, get **5** as the sum and set it down as the middle part of the answer ; and

(iii) We multiply **2** and **3** vertically, get **6** as their product and put it down as the last (the right-hand-most,) part of the answer.

Thus **12×13=156.**

A few other examples may also be tested and will be found to be correct :—

(1) 12
11

1 : 1+2 : 2
=132

(2) 16
11

1 : 1+6 : 6
=176

(3) 21
14

2 : 8+1 : 4
=294

(4) 23
21

4 : 2+6 : 3
=483

(5) 41
41

16 : 4+4 : 1
=1681

Note :—When one of the results contains more than 1 digit, the right-hand-most digit thereof is to be put down there and the preceding (i. e. left-hand-side) digit (or digits) should be carried over to the left and placed under the previous digit (or digits) of the upper row until sufficient practice has been achieved for this operation to be performed mentally. The digits carried over may be shown in the working (as illustrated below) :—

(1)	(2)	(3)	(4)	(5)	(6)
15	25	32	35	37	49
15	25	32	35	33	49
105	40	924	905	901	1621
12	225	1	32	32	78
225	625	1024	1225	1221	2401

The **Algebraical ptinciple** involved is as follows :—

Suppose we have to multiply $(ax+b)$ by $(cx+d)$. The product is $acx^2+x(ad+bc)+bd$. In other words, the first term (i. e. the coefficientof x^2) is got by vertical multiplication of a and c ; the middle term (i. e. the coefficient of x) is obtained by the cross-wise multiplication of *a* and *d* and of *b* and *c* and the addition of the two products ; and the **independent** term is arrived at by vertical multiplication of the absolute terms. And, as all arithmetical numbers are merely algebraic expres-

6

sions in **x** (with **x=10**), the algebraic principle explained above is readily applicable to arithmetical numbers too. Now, if our multiplicand and multiplier be of **3** digits each, it merely means that we are multiplying (ax^a+bx+c) by (dx^a+ex+f) (where **x=10**) :—

ax^a+bx+c
dx^2+ex+f

$adx^4+x^3\ (ae+bd)+x^a\ (af+be+cd)+x\ (bf+ce)+cf$

We observe here the following facts :—

(i) that the coefficient of x^4 is got by the vertical multiplication of the first digit (from the left side) ;

(ii) that the coefficient of $x^3$is got by the cross-wise multiplication of the first two digits and by the addition of the two products ;

(iii) that the coefficient of x^a is obtained by the multiplication of the first digit of the multiplicand by the last digit of the multiplier, of the middle one by the middle one and of the last one by the first one and by the addition of all the **3** products ;

(iv) that the coefficient of x is obtained by the cross-wise multiplication of the second digit by the third one and conversely and by the addition of the two products ; and

(v) that the independent term results from the vertical multiplication of the last digit by the last digit.

We thus follow a process of ascent and of descent (going forward with the digits on the upper row and coming rearward with the digits on the lower row). If and when this principle (of ordinary Algebraic multiplication) is properly understood and carefully applied to the Arithmetical multiplication on hand (where x stands for 10), the *Ūrdhva* Tiryak *Sūtra* may be deemed to have been successfully mastered in actual practice.

A few illustrations will serve to illustrate this *Ūrdhva-Tiryak* process of vertical and cross-wise multiplications :—

(1) 111 111 ——— 12321	(2) 108 108 ——— 10 60 4 1 6 ——— 11 66 4	(3) 109 111 ——— 11 099 1 ——— 12 099
(4) 116 114 ——— 12 1 0 4 1 1 2 ——— 1 3 2 2 4	(5) 116 116 ——— 12 32 6 1 13 ——— 13 45 6	(6) 582 231 ——— 10 1 3 42 3 3 1 ——— 13 4 4 42
(7) 532 472 ——— 20 7 9 04 4 3 2 ——— 25 1 1 04	(8) 785 362 ——— 2 1 6 7 6 0 6 7 4 1 ——— 2 8 4 1 7 0	(9) 321 52 ——— 0 5 692 1 1 ——— 1 6 692
(10) 795 362 ——— 2 1 9 3 8 0 6 8 4 1 ——— 2 8 7 7 9 0	(11) 1021 2103 ——— 2147163	(12) 621 547 ——— 30 4 5 87 3 5 1 ——— 33 9 6 87
(13) 6471 6212 ——— 36 6 6 6 752 3 5 3 1 1 ——— 40 1 9 7 852	(14) 8 7 2 6 5 3 2 1 1 7 ——— 2 4 7 8 7 2 7 5 7 5 3 2 3 9 6 2 4 3 ——— 2 8 0 2 6 9 0 0 0 5	

N.B.—It need hardly be mentioned that we can carry out this (*Ūrdhva-Tiryak*) process of multiplication from left to right or from right to left (as we prefer). All the diffe-

rence will be that, in the former case, two-line multiplication will be necessary (at least mentally) while, in the latter case, one-line multiplication will suffice., (but careful practice is necessary).

Owing to their relevancy to this context, a few Algebraic examples (of the *Ūrdhva-Tiryak* type) are being given.

(1) $a+b$
$a+9b$

$a^2+10\,ab+9b^2$

(2) $a+3b$
$5a+7b$

$5a^2+22ab+21b^2$

(3) $3x^2+5x+7$
$4x^2+7x+6$

$12x^4+41x^3+81x^2+79x+42$

(4) $x^5+3x^4+5x^3+3x^2+x+1$
$7x^5+5x^4+3x^3+x^2+3x+5$

$7x^{10}+26x^9+53x^8+56x^7+43x^6+40x^5+41x^4+38x^3+19x^2$
$+8x+5$

Note :—If and when a power of x is absent, it should be given a zero coefficient ; and the work should be proceeded with exactly as before. For example, for $(7x^3+5x+1)(3x^3+x^2+3)$, we work as follows :—

$7x^3+0+5x+1$
$3x^3+x^2+0+3$

$21x^6+7x^5+15x^4+29x^3+x^2+15x+3$

The use of the ***Vinculum*** :

It may, in general, be stated that **multiplications** by digits higher than **5** may some times be facilitated by the use of the **vinculum.** The **following** example will illustrate this :—

(*1*) 576	(2) OR $6\overline{24}$	But the vinculum process is
214	214	one which the student must
———		very carefully **practise,** before
109944	122446	he resorts to it and relies
1332	+111	on it.
123264	123264	

Miscellaneous Examples :

There being so many methods of multiplication one of them (the ***Ūrdhva-Tiryak*** one) being perfectly general and therefore applicable to all cases and the others (the ***Nikhilaṁ*** one, the ***Yāvadūnam*** etc.) being of use in certain special cases only, it is for the student to think of and weigh all the possible alternative processes available, make up his mind as to the simplest method in each particular case and apply the formula prescribed therefor.

We now conclude this chapter with a number of miscellaneous examples and with our own "running comments" thereon giving the students the necessary experience for **making the best** possible selection from amongst the various alternative method, in question :—

(1) 73x37

(i) By ***Ūrdhva-Tiryak*** rule,

```
   73
   37
 ----
 2181
   52
 ----
=2701
```

or (ii) by the same method but with the use of the vinculum.

$$
\begin{array}{r}
1\ \bar{3}\ 3 \\
0\ 4\ \bar{3} \\
\hline
0\ 4\ \bar{5}\ 1\ \bar{9} \\
\bar{1}\ \bar{2}\ \ \ \ \ \\
\hline
=\ 2\ 7\ 0\ 1
\end{array}
$$

Evidently, the former is better.

(2) 94x81

(i) By *Ūrdhava-Tiryak,*

94
81
———
7214=7614
 4
——

(ii) Or $1\bar{1}4$
$1\bar{2}1$
———
$1\bar{3}7\bar{9}4$=7614
———

(ii) By ibid (with the use of the Vinculum)

Evidently the former is better ; but

Or

(iii) The Nikhilam Method is still better :—81—19
94— 6
———
$75/{}_{1}14$=7614

---

(3) 123×89

(i) 123
089
———
08527
242
———
=10947

Or (ii) 123
111
———
$110\overline{53}$
=10947

Or (iii) 123+23
89—11
———
112/253
———
110/53=109/47

---

(4) 652×43

(i) 652
043
———
04836
232
———
28036

(ii) The Vinculnm method is manifestly cumbrous in this case and need not be worked out.

($1\overline{35}2$
×0043)
———

(iii) The *Nikhilam* method may be used and will he quite easy; but we will have to take a multiple of 43 which will bring it very near 1000. Such a multiple is 43×23=989 ; and we can work with it and finally divide the whole thing out by 23. This gives us the same answer (23/036).

652—348
989—011
————
$641/{}_{3}828$
23) 644/828
———
28/036

Therefore, the *Ūrdhva* (general) process is obviously the best (in this case).

(5) 123x112 (*Nikhilam*)

(i)	(ii)	(iii)
123 112 —— 13276 6 —— =13776	As all the digists are within 5, the Vinculum method is manifestly out of place.	123+23 112+12 —— 135 / $_2$76 =137 176

Both the first and the third methods seem equally good.

(6) 99×99

(i)	(ii)	(iii)	(iv)
99 99 —— 8121 168 —— =9801	$10\bar{1}$ $10\bar{1}$ —— $10\bar{2}01$ =9801	99—1 99—1 —— =98/01 appropriate	The (*Yāvadūnam*) method is also quite appropriate & easy 99^2=98/01

(*7*)	(*8*)	(*9*)	(*10*)
246 131 —— 20026 122 —— =32226	222 143 —— 20646 111 —— =31746	642 131 —— 62002 221 —— =84102	321 213 —— 67373 1 —— =68373

(*In all these 4 cases (Nos. 7—10), the General formula fits in at once*).

(*11*) 889×898

(i)	Or (ii)	*Or* (iii)	
889 898 —— 646852 13047 21 —— =798322	$1\bar{1}\bar{1}\bar{1}$ $1\bar{1}0\bar{2}$ —— $1\bar{2}0\bar{2}322$ =798/322	889— 111 898— 102 787 / $_{11}$322 =798 / 322	*111+11 102+ 2 —— 113 / 22 ——

Note ;—Here in (iii) *Nikhilam* method, the vertical multiplication of 111 and 102 is also performed in the same manner (as shown in the *marked margin).

(12) (i) 576 × 328

```
   576
  x328
 ------
 151288
   3764
 ------
=188928
```

Or (ii) Vinculum method inappropriate

Or (iii)

```
  576—424
  984— 16
 ----------
3)560 /₆784
 ----------
=188 / 928
```

N.B. 984 being 3×328, we have made use of it & then divided out by 3

(13) 817x332

(i)

```
   817
   332
 ------
 247034
   2421
 ------
=271244
```

Or (ii) Vinculum method may also be used.

Or (iii) ∴ 332×3=996

```
∴ 817—183
   9964
  -------
 3)813/732
  -------
  =271 /244
```

(14) 989x989

(i)

```
   989
   989
 ------
 814641
  14248
     21
 ------
=978121
```

Or (ii) Vinculum method also useful—

$$10\bar{1}\bar{1}$$
$$10\bar{1}\bar{1}$$
$$\overline{\quad 1022121 \quad}$$
$$=978/121$$

Or (iii)

```
 989— 11
 989— 11
 --------
=978 / 121
```

Or (iv) (*Yāvadūnam*). 989^2=978/121 This is the best.

(15) 8989 X 8898

(i)

```
    8989
    8898
 --------
64681652
 1308147
    2221
 --------
=79984122
```

Or (ii)

$$1\ \bar{1}\ 0\ \bar{1}\ \bar{1}$$
$$1\ \bar{1}\ \bar{1}\ 0\ \bar{2}$$
$$\overline{\ 1\ 2\ 0\ 0\ 2\ 4\ 1\ 2\ 2\ }$$
$$=7\ 9\ 9\ 8\ 4\ 1\ 2\ 2$$

Or (iii)

```
 8989—1011
 8898—1102
 ----------
 7887 / 4122*
  111 /
 ----------
 7998 / 4122
```

```
 *1011+ 11
  1102+102
 ----------
  1113/₁122
 =1114/ 122
```

(16) 213×213

(i)

```
  213
  213
 -----
 44369
 1
 -----
=45369
```

Or (ii) Vinculum method not suitable.

Or

```
 213+ 13
 213+ 13
 --------
 226 / ₁69
 ×2 /
 --------
 452 / ₁69
=453 /  69
```

N. B. The digits being small, the general formula is always best.

PRACTICAL APPLICATION

IN

"COMPOUND MULTIPLICATION"

A. *Square Measure, Cubic Measure Etc.*

This is not a separate subject, all by itself. But it is often of practical interest and importance, even to lay people and deserves our attention on that score. We therefore deal with it briefly.

Areas of Rectangles.

Suppose we have to know the area of a Rectangular piece of land whose length and breadth are 7′ 8″ and 6′ 11″ respectively.

According to the conventional method, we put both these measurements into uniform shape (either as inches or as vulgar fractions of feet-preferably the latter) and say :—

$$\text{Area}=\frac{92}{12}\times\frac{71}{12}=\frac{6532}{144}=\frac{1633}{36}$$

```
36) 1633 (45 sq. ft
    144
    ---
     193
     180
     ---
      13
    X144
    ----
36) 1872 (52 sq. in.
    36
    --
     72
     72
     --
```

∴ **Area=45** sq. ft. 52 sq. in.

In the Vedic method, however, we make use of the **Algebraical** multiplication and the ***Ādyam Sūtra*** and say :—

$$\left.\begin{array}{r}\text{Area}=5x+11\\ \times 7x+8\end{array}\right\}=35x^2+117x+88$$

Splitting the middle term (by dividing by **12**), we get 9 and 9 as **Q** and **R**

$\therefore E = 35x^2 + (9 \times 12 + 9)x + 88$
$= 44x^2 + (9 \times 12)x + 88$
$= 44$ sq. ft. + 196 sq. in.
$= 45$ sq. ft. + 52 sq. in.

And the whole work can be done mentally :

(2) Similarly 3' 7" × 5'10" $\left.\right\}$ $= 15x^2 + 66x + 70$
$= 20x^2 + (5 \times 12) + 70$
$= 20$ sq. ft. + 130 sq. in.

and (3) $7x + 11$ × $5x + 8$ $\left.\right\}$ $= 35x^2 + 111x + 88$
$= 44$ sq. ft. + 124 sq. in.

Volumes of Parallelepipeds:

We can extend the same method to sums relating to 3 dimensions also. Suppose we have to find the volume of a parallelepiped whose dimensions are 3' 7", 6' 10" and 7' 2".

By the customary method, we will say :—

$C.C = \frac{43}{12} \times \frac{70}{12} \times \frac{86}{12}$ (with all the big multiplication and divisions involved). But, by the Vedic process, we have :—

$\left.\begin{array}{l} 3x+7 \\ 5x+10 \end{array}\right\} = 20x^2 + 10x + 10$
$7x + 2$

$140x^3 + 110x^2 + 90x + 20$
$= 149x^3 + 9x^2 + 7x + 8$
$= 149$ cub. ft and 1388 cub. in.

Thus, even in these small computations, the customary method seems to have a natural or ingrained bias in favour of needlessly big multiplications, divisions, vulgar fractions etc., etc., for their own sake. The Vedic Sūtras, however, help us to avoid these and make the work a pleasure and not an infliction.

PRACTICE AND PROPORTION

IN

COMPOUND MULTIPLICATION.

The same procedure under the ऊर्ध्वतिर्यक् (*Ūrdhva-Tiryak Sūtra*) is readily applicable to most questions which come under the headings "Simple Practice" and "Compound Practice", wherein "ALIQUOT" parts are taken and many steps of working are resorted to under the current system but wherein according to the Vedic method, all of it is mental Arithmetic,

For example, suppose the question is :—

"In a certain investment, each rupee invested brings Rupees two and five annas to the investor. How much will an outlay of Rs. 4 and annas nine therein yield ?"

THE FIRST CONVENTIONAL METHOD.

By Means of Aliquot Parts.

	Rs. As. Pa.
For One Rupee	2—5—0
For 4 Rupees	9—4—0
8 As.=$\frac{1}{2}$ of Re. 1	1—2—6
1 a=) of 8 As.	0—2—3$\frac{3}{4}$
Total for Rs. 4 and annas 9.	10—8—9$\frac{3}{4}$

Second Current Method.

(By Simple Proportion)

Rs. 2—5—0=$\frac{87}{16}$;

and Rs. 4—9—0=Rs. $\frac{78}{16}$

∵On Re 1, the yield is Rs. $\frac{87}{16}$

∴On Re. $\frac{78}{16}$, the yield is Rs. $\frac{87}{16}$ X $\frac{78}{16}$=Rs. $\frac{2701}{256}$

```
266) 2701 (10—8—98
     256
     ----
      141
     ×16
     ----
266) 2045 ( 8
     2304
     ----
      208
     ×12
256) 2496 ( 9
     2304
     ----
      192
     ----   =3/4
      256
```

By the Vedic one-line **method :**

```
2x + 5
4x + 9
-----------
8x² / 38x/45
```

Splitting the middle term (or by simple division from right to left) :

$10x^2+6x+2\frac{13}{16}$

$=$Rs. 10 and $8\frac{13}{16}$ annas

A few more instances may be taken :—

(1) **Rs. 2/5 × Rs. 2/5**

```
  2—5
  2—5
---------
4/20/25
```

4/20/25 = Rs. 5/5$\frac{9}{16}$ annas

(2) **Rs. 4/9** × Rs. **4/9**

```
  4—9
  4—9
---------
16/72/81
```

16/72/81 = Rs. 20/13$\frac{1}{16}$ annas

(3) Rs. **16/9 × Rs. 16/9**

```
  16—9
  16—9
-----------
256/288/81
```

256/288/81 = Rs. 274/ annas 5$\frac{1}{16}$

(4) **Rs.** 4/13 X **Rs.** 4/13

(i) By the current 'Practice' method

	Rs.—as
For Re. 1	4—13
For Rs. 4	19— **4**
8 **annas**=$\frac{1}{2}$ of Re I	2— **6$\frac{1}{2}$**
4 annas=$\frac{1}{2}$ of 8 As.	1— **3$\frac{1}{4}$**
1 **a**=$\frac{1}{4}$ of 4 **annas**	0— **4$\frac{13}{16}$**
Total	23— **2$\frac{9}{16}$**

(ii) By the current 'Proportion' method.

Rs. 4/13 = Rs. $\frac{77}{16}$

$\therefore \frac{77}{16} \times \frac{77}{16} = \frac{5929}{256}$

256) 5929 **(23—2$\frac{9}{16}$**
612
- 809
768
41
×16
256) 656 (2
512
144
256 **=9/16**

(iii) By the one-line Vedic Method.

4—13
4—13

16/104/169=Rs. 23/2$\frac{9}{16}$ annas

N.B.—**Questions relating** to paving, carpeting, ornamenting etc., **etc.** (which are-under the current system usually dealt with by the **'Practice'** method or by the 'Proportion' process) can all be readily answered by the *Urdhva-Tiryak* method.

For example, **suppose** the question is :—

At the rate of 7 **annas 9** pies per foot, what will be the ost for 8 yards 1 foot 3 inches ?

25—3
7—9

175/246/27
=**195** annas **8¼** pies
=**Rs.** 12/3/83

CHAPTER IV

DIVISION (*by* the *Nikhilam Method*).

Having dealt with Multiplication at fairly considerable length, we now go on to **Division**; and there we start with the *Nikhilam* method (which is a special one).

Suppose we have to divide a number of dividends (pf two digits each) successively by the same Divisor 9 we make a chart therefor as follows:—

(1) 9) 1/2	(2) 9) 2/1	(3) 9) 3/3
/1	/2	/3
1/3	2/3	3/6
(4) 9) 4/0	(5) 9) 5/2	(6) 9) 6/1
/4	/5	/6
4/4	5/7	6/7
(7) 9) 7/0	(8) 9) 8/0	
/7	/8	
7/7	8/8	

Let us first split each dividend into a left-hand part for the Quotient and a right-hand part for the Remainder and divide them by a vertical line.

In all these particular cases, we observe that the **first digit** of the Dividend becomes the Quotient and the sum of the two digits **becomes** the Remainder. This means that we can mechanically take the **first** digit down for the **Quotient-column** and that, by adding the quotient to the **second** digit, we can get the Remainder.

Next, we take as Dividends, another set of bigger numbers of 3 digits each and make a chart of them as follows:—

(1) 9) 10/3	(2) 9) 11/3	(3) 9) 12/4
1/1	1/2	1/3
11/4	12/5	13/7
(4) 9) 16/0	(5) 9) 21/1	(6) 9) 31/1
1/7	2/3	3/4
17/7	23/4	34/5

In these cases, we note that the Remainder and the sum of the digits are still the same and that, by **taking** the **first** digit of the Dividend down **mechanically** and adding it to the second digit of the dividend, we get the second digit of the quotient and that by adding it to the third digit of the dividend, we obtain the remainder.

And then, by extending this procedure to still bigger numbers (consisting of still more digits), we are able to get the quotient and remainder correctly. For example,

(1)	9)	1203/1	(2)	9)	1230/1	(3)	9)	120021/2
		133/6			136/6			13335/6
		1336/7			1366/7			133356/8

And, thereafter, we **take a few** more oases as follows:—

(1)	9)	1/8	(2)	9)	22/5	(3)	9)	13/6
		/1			2/4			1/4
		1/9			24/9			14/10
(4)	9)	23/7						
		2/5						
		25/12						

But in all these cases, we find that the Remainder is the same as or greater than the Divisor. As this is not permissible, we re-divide the Remainder by 9, carry the quotient over to the Quotient column and **rctain** the final Remainder in the Remainder **cloumn**, as follows:—

(1)	9)	1/S	(2)	9)	22/5	(3)	9)	13/6
		/1			2/4			1/4
		1/9			24/9			14/10
		2/0			25/0			15 / 1
(4)	9)	23/7						
		2/5						
		25/12						
		26 / 3						

We also notice that, when the Remainder is greater than the Divisor, we can do the consequent final Division by the same method, as follows :—

```
(1)  9) 13/6          (2) 9) 23/7          (3) 9) 101164/9
        1/4                  2/5                  11239/13
       ------               ------               -----------
       14/1/0               25/1/2               1123913/2/2
           /1                   /1                        /2
       ------               ------                      ----
          1/1                  1/3                       2/4
       ------               ------               -----------
        15 / 1               26 / 3              112405   /4
```

We next take up the next lower numbers (8, 7 etc.) as our Divisors and note the results, as follows :—

```
(1)  8)  2/3          (2) 7)  1/2          (3)  6)  1/1
        -----               -----                 -----
          /4                  /3                    /4
        -----               -----                 -----
         2/7                 1/5                   1/5
```

Here we observe that, on taking the first digit of the Dividend down mechanically, we do not get the Remainder by adding that digit of the quotient to the second digit of the dividend but have to add to it twice. thrice or 4 times the quotient-digit already taken down. In other words, we have to multiply the quotient-digit by 2 in the case of 8, by 3 in the case of 7, by 4 in the case of 6 and so on. And this again means that we have to multiply the quotient-digit by the Divisor's complement from 10.

And this suggests that the *Nikhilaṁ* rule (about the subtraction of all from 9 and of the last from 10) is at work ; and, to make sure of it, we try with bigger divisions, as follows:

```
(1)  89)  1/11       (2)  73) 1/11       (3)  888) 1/234
     --                   --                  ---
     11    /11            27   /27            112   /112
          -----               -----                 -----
          1/22                1/38                  1/346
```

(4)
8888) 1/2345
1112 /1112
1/3457

(5)
7999) 1/2345
2001 /2001
1/4346

(6)
8897) 1/2345
1103 /1103
1/3448

---

(7)
8897) 1/1203
1103 /1103
1/2306

(8)
7989) 1/0102
2011 /2011
1/2113

(9)
899997) 1/010101
100003 /100003
1/110104

---

(10)
89) 11/11
11 1/1
/22
12/43

(11)
89) 100/13
11 11/
1/1
/22
112/45

(12)
882) 12/345
112 1/12
/336
13/801

---

(13)
8997) 21/0012
1003 2/006
/3009
23/3081

(14)
8998) 30/0000
1002 3/006
/3006
33/3066

(15)
8888) 10/1020
1112 1/112
/1112
11/3252

---

(16)
8987) 20/0165
1013 2/026
/2026
22/2451

(17)
899) 10/102
101 1/01
/101
11/213

(18)
89998) 20/02002
10002 2/0004
/20004
22/22046

---

(19)
89997) 10/10101
10003 1/0003
/10003
11/20134

(20)
89997) 12/34567
10003 1/0003
/30009
13/64606

(21)
88987) 10/30007
01013 0/1013
/00000
10/40137

---

(22)
99979) 111/11111
00021 00/021
0/0021
/00021
111/13442

(23)
88) 110/01
12 12/
2/4
/48
124/89
=125/1

In all the above examples, we have deliberately taken as **Divisiors,** numbers containing big digits. The reason therefor is as follows:—

(i) It is in such division (by big **divisions)** that the student **finds** his chief difficulty, because he **has** to multiply long big numbers by the "trial" digit of the quotient at every step and subtract that result from each dividend at each step; but, in our method (of the ***Nikhilam*** formula), the bigger the digits, the smaller will be the required complement (from **9** or **10** as the case may be); and the multiplication-task is lightened thereby.

(ii) There is no subtraction to be done at all!

(iii) And, even as regards the multiplication, we have no **multiplication** of numbers by numbers as such but only of a single digit ***by*** a single digit, with the pleasant consequence that, at no stage, is a student called upon to multiply more than **9** by more than 9. In other words, **$9\times9=81$, is** the utmost multiplication he has to perform.

A single sample example will suffice to prove **this**:

(24)	9819)	2	**01 37**
	$\overline{0181}$		**02162**
		2	**0499**

Note:—In this case, the product of 8 and 9 is written **down in** its **proper place,** as **16** (with no **"carrying"** over to the left) and so on.

Thus, in our **"division"—process** (by the ***Nikhilam*** formula), we perform only small **single-digit** multiplications; we do *no* subtraction and *no* division at all; and yet we readily obtain the required quotient and the required Remainder. In fact, we have accomplished our division-work in full, without actually doing any division at all!

As for divisors consisting of small digits, another simple formula will serve our purpose and is to be dealt with in the next chapter. Just at present (in this chapter), we deal only with big divisors and explain how simple and easy such difficult multiplications can be made (with the aid of the ***Nikhilam Sūtra***).

And herein, we take up a few more illustrative examples relating to the cases (already referred to) wherein the Remainder exceeds the Divisor and explain the process, by which this difficulty can be easily surmounted (by further application of the same ***Nikhilam*** method):—

```
(25)   88) 1   98
       12      12
           -------
           1  110
         ---------
```

The Remainder here (**110**) being greater than the Divisor (88) we have to divide **110** by 88 and get the quotient and the final remainder and carry the former over and add it to the quotient already obtained. Thus, we say :—

```
88) 1   10
12      12
    ------
    1   22
    ------
```

so, we add the newly obtained **1** to the previously obtained **1**; and put down 2 as the quotient and 22 as the Remainder.

This double process can be combined into one as follows :—

```
88) 1    98
12       12
    -------
    1  1/10
        /12
    1  1/22
    -------
    2   /22
    -------
```

A few more illustrations will serve to help the student in practising this method :—

```
(26)  89997)  12        94567        (27)  97)  1        98
      10003    1        0003               03            03
                        30009                       ----------
             ------------------                     1   1   01
              13   1    24606                                03
                        10003                       ----------
             ------------------                     2       04
              14        34609                       ----------
             ------------------
```

```
(28)  99979)  111       99171
      00021    00       021
                0       0021
                        00021
             ------------------
              111   1   01502
                        00021
             ------------------
              112       01523
             ------------------
```

Thus, even the whole lengthy operation (of division of **11199171** by **99979**) involves *no* division and *no* subtraction and consists of a few multiplications of single digits by single digits and a little addition (of an equally easy character).

Yes ; this is all good enough so far as it **goes** ; but it provides only **for** a particular type (namely, of divisions involving large-digit numbers). Can it help us in other divisions (i.e. those which involve small-digit divisors)?

The answer is a candidly emphatic and unequivocal No. An actual sample specimen will prove this :—

Suppose we have to divide 1011 by 23. By the *Nikhilam* method, the working will be as follows:—

(29)	23)	10		1		1	
	$\overline{77}$	7		7			
			4	9	4	9	
		17	6	2		0	
				42		42	
		23	4	8		2	
				28		28	
		27	3	9		0	
				21		21	
		30	3	2		1	
				21		21	
		33	2	5		2	
				14		14	
		35	2	0		6	
				14		14	
		37	1	6		0	
				7		7	
		38	1	3		7	
				7		7	
		3	9	1	1	4	
		4		—9		2	(4 times the divisor)
		43		2		2	

This is manifestly not only too long and cumbrous but much more so than the current system (which, in this particular case, is indisputably shorter and easier).

In such a case, we can use a multiple of the divisor and finally multiply again (by the *Ānurūpya* rule). Thus,

(30) 23×4=92)	10	11
$\overline{0}\overline{8}$	0	8
	10	91
	×4	
	40	−69
	43	22

But even this is too long and cumbrous; and this is a suitable case for the application of the परावर्त्य (*Parāvartya*) method. This we proceed to explain in the next chapter.

CHAPTER V

DIVISION (by the *Parāvartya method)*

We have thus found that, although admirably suited for application in the special or particular cases wherein the divisor-digits are big ones, yet the *Nikhilam* method does not help us in the other cases (namely, those wherein the divisor consists of small digits). The last example (with 23 as divisor) at the end of the last chapter has made this perfectly clear. Hence the need for a formula which will cover the other cases. And this is found provided for in the *Parāvartya Sūtra,* which is a special-case formula, which reads "*Parāvartya Yojayet*" and which means "Transpose and apply".

The well-known rule relating to transposition enjoins invariable change of sign with every change of side. Thus+ becomes− and conversely ; and × becomes ÷ and conversely. In the current system, this law is known but only in its application to the transposition of terms from left to right and conversely and from numerator to denominator and conversely (in connection with the solution of equations, the proving of Identities etc., etc ; and also with regard to the Remainder Theorem, Horner's process of Synthetic Division etc. etc.) According to the Vedic system, however, it has a number of applications, one of which is discussed in the present chapter.

At this point, we may make a reference to the Remainder Theorem and Horner's process and then pass on to the other most interesting applications of the *Parcivartya Sara.*

***The Remainder Theorem*:**

We may begin this part of this exposition with a simple proof of the Remainder Theorem, as follows:

If E, D, Q & R be the Dividend, the Divisor, the Quotient and the Remainder in a case of division and if the divisor is (x−p), we may put this relationship down algebraically as follows :—

$$E=DQ+R \text{ i.e. } E=Q(x-p)+R.$$

And if we put **x=p,** x—p becomes zero; and the Identity takes the shape, **E=R.** In other words, the given expression E itself (with p substituted for x) will be the Remainder.

Thus, the given expression E (i.e. the Dividend itself) (with p substituted for x) **automatically** becomes the remainder. And p is automatically available by putting **x—p=0 i.e.** by merely reversing the sign of the—p (which is the absolute term in the binomial divisor). In general terms, this means that, if c be $ax^n+bx^{n-1}+cx^{n-2}+dx^{n-3}$ etc. and if D be x—p, the remainder is $ap^n+bp^{n-1}+cp^{n-2}+dp^{n-3}$ and so on (i.e. E with p substituted for x). This is the Remainder Theorem.

Horner's process of Synthetic Division carries this still further and tells us the quotient too. It is, however, only a very small part of the ***Parāvartya*** formula (which goes much farther and is capable of numerous applications in other directions also).

Now, suppose we have to divide $(12x^2-8x-32)$ by (x—2).

x—2	$12x^2-8x$	-32
2	24	$+32$
	$12x+16$	0

We put x—2 (the Divsior) down on the left (as usual); just below it, we put down the—2 with its sign changed; and we do the multiplication work just exactly as we did in the previous chapter.

A few more algebraic examples may also be taken :—

(1) Divide $7x^2+5x+3$ by **x—1**

x—1	$7x^2+5x$	$+3$
1	7	$+12$
	$7x+12$	15

∴The quotient is **7x+12**; and the Remainder is **15.**

(2)

x+1	$7x^2+5x$	$+3$
—1	-7	$+2$
	$7x-2$	$+5$

(3)

x—2	x^3+7x^2+6x	$+5$
2	$2+18$	$+48$
	$x^2+9x+24$	53

(4) $\frac{x-3}{3}$

x^3-x^2+	$7x$	$+3$	
	$3+6$		39
x^2+2	$+13$		42

(6) $\frac{x-6}{5}$

x^3-3x^2+10x		-7
6	$10	100
x^2+2	$+20$	$+93$

At this stage, the student should practise the whole process as a MENTAL exercise (in respect of binomial divisors at any rate). For example, with regard to the division of $(12x^2-8x-32)$ by the binomial $(x-2)$, one should be able to say :—

$$\frac{12x^2-8x-32}{x-2}=12x+16 \text{ ; and } R=O$$

The procedure is as follows :—

(i) $\frac{12x^2}{x}$ gives 12 as the first coefficient in the quotient ; and we put it down;

(ii) multiply 12 by -2, reve-se the sign and add to the next coefficient on the top (numerator). Thus $12 \times -2=-24$, Reversed, it is 24. Add-8 and obtain 16 as the next coefficient of the Quotient.

(iii) multiply 16 by-2 ; reverse the sign and add to the next coefficient on top. Thus $16 \times -2=-32$; Reversed, it is 32; add-32 and obtain 0 as the Remainder.

Similarly, (1) $\frac{7x^2+5x+3}{x-1}$ $\therefore Q=7x+12$; and $R=15$

(2) $\frac{7x^2+5x+3}{x+1}$ $\therefore Q=7x-2$; and $R=5$

(3) $\frac{x^3+7x^2+6x+5}{x-2}$ $\therefore Q=x^2+9x+24$; and $R=53$

(4) $\frac{x^3-x^2+7x+3}{x-3}$ $\therefore Q=x^2+2x+13$; and $R=42$

and (5) $\frac{x^3-3x^2+10x-7}{x-5}$ $\therefore Q=x^2+2x+20$; $R=93$

This direct and straight application of the *Parāvartya Sūtra* should be so well practised as to become very simple MENTAL arithmetic. And the student should be able to say at Once :—

(6) $\frac{x^3+7x^2+9x+11}{x-2}$ $\therefore Q=x^2+9x+27$; $R=65$;

and (7) $\frac{x^4-3x^3+7x^2+5x+7}{x-4}$ $\therefore Q=x^3+x^2+11x+49$; and $R=203$.

Extending this process to the case of divisors containing three terms, we should follow the same method, but should also take care to reverse the signs of the coefficient in all the other terms (except the first) :—

(1)
$$\begin{array}{l|lll}
x^2-x-1 & x^4-x^3+x^2 & & +3x+5 \\
\overline{\ \ 1+1} & \quad\ 1+1 & & \\
 & \qquad\ 0 & +0 & \\
 & & \ 2 & +2 \\
\hline
 & x^2+0+2 & & +5x+7
\end{array}$$

$\therefore Q=x^2+2$; and $R=5x+7$

(2)
$$\begin{array}{l|lll}
x^2-2x-9 & 6x^4+13x^3+39x^2 & & +37x+45 \\
\overline{\ \ 2+9} & \quad 12 \ \ +54 & & \\
 & \qquad\ 50 & +225 & \\
 & & & 286+1287 \\
\hline
 & 6x^2+25 \quad +143 & & +548+1332
\end{array}$$

$\therefore Q=6x^2+25x+143$; and $R=548x+1332$

(3)
$$\begin{array}{l|lll}
x^2+1 & 2x^4-3x^3+0 & & -3x-2 \\
\overline{\ \ 0-1} & \quad 0-2 & & \\
 & \qquad 0 & +3 & \\
 & & \ 0 & +2 \\
\hline
 & 2 \ \ -3 \ \ -2 & & 0+0
\end{array}$$

$\therefore Q=2x^2-3x-2$ and $R=O$

Note the zero x^2 and the zero x carefully.

(4) $\frac{x^2-2x+1}{2-1}$ $\quad x^3-3x^2 \quad +3x-1$
$\qquad\qquad\qquad\quad 2 \qquad -1$
$\qquad\qquad\qquad\qquad\quad -2+1$

$1 \; -1 \qquad 0+0 \qquad \therefore Q=x-1$; and $R=0$

(5) $\frac{x^2+2x+4}{-2-4}$ $\quad 2x^3+9x^2 \quad +18x+20$
$\qquad\qquad\qquad\quad -4 \qquad -8$
$\qquad\qquad\qquad\qquad\quad -10-20$

$2 \; +5 \qquad 0+0 \qquad \therefore Q=2x+5$; & $R=0$

(6) $\frac{x^8-x^S+2x-3}{1-2+3}$ $\quad x^5+0+x^3 \quad -7x^2+0+9$
$\qquad\qquad\qquad\quad 1-2 \quad +3$
$\qquad\qquad\qquad\qquad 1 \quad -2 \; +3$
$\qquad\qquad\qquad\qquad\quad 0 \; +0+0$

$1+1+0 \qquad -6 \; +3+9$

Note the zero x^4 and the zero x carefully.

$\therefore Q=x^2+x$; and $R=-6x^2+3x+9$

(7) $\frac{x^2-x+1}{1-1}$ $\quad x^4+0+x^2 \quad +0+1$
$\qquad\qquad\qquad\quad 1-1$
$\qquad\qquad\qquad\quad 1 \quad -1$
$\qquad\qquad\qquad\qquad\quad 1-1$

$1+1+1 \qquad 0+0 \qquad \therefore Q=x^2+x+1$; and $R=0$

Note the zero x^3 and zero x carefully.

(8) $\frac{x^a-2x^2+1}{2+0-1}$ $\quad -2x^5-7x^4+2x^3 \quad +18x^a \; -3x-8$
$\qquad\qquad\qquad\quad -4 \; +0 \quad +2$
$\qquad\qquad\qquad\qquad -22 \quad +0 \; +11$
$\qquad\qquad\qquad\qquad\quad -40 \; +0+20$

$-2 \; -11 \; -20 \qquad -20 \; +8+12$

Note the zero x in the Divisor carefully

$\therefore Q=-2x^a-11x-20$; and $R=-20x^2+8x+12$

In all the above oases, the first coefficient in the divisor happened to be 1 ; and therefore there was no risk of fractional coefficients coming in. But what about the cases wherein, the first coefficient not being unity, fractions will have to be reckoned with ?

The answer is that all the work may be done as before, with a simple addition to the effect that every coefficient in the answer must be divided by the first coefficient of the Divisor.

Thus, $\frac{2x-4}{4}$ $\quad -4x^3-7x^2+9x-12$

$\qquad\qquad -8 \quad -30 \quad -42$

$\qquad -2 \quad -15/2 \quad -21/2 \quad -54$

$\therefore Q=-2x^2-7\frac{1}{2}x-10\frac{1}{2}$; and $R=-54$

This, however, means a halving of each coefficient (at every step); and this is not only more cumbrous but also likely to lead to confusion, reduplication etc.

The better method therefore would be to divide the Divisor itself at the very outset by its first coefficient, complete the working and divide it all off again, once for all at the end. Thus :—

$\frac{2x-4}{\frac{x-2}{2}}$ $\quad -4x^3-7x^2+9x \quad -12$

$\qquad -8 \quad -30 \quad -42$

$2)-4 \quad -15 \quad -21 \quad -54$

$-2 \quad -7\frac{1}{2} \quad -21/2 \quad -54$

N.B.—Note that the R always remains constant.

Two more illustrative examples may be taken :—

(1) $3)\frac{3x-7}{\frac{x-2\frac{1}{3}}{2\frac{1}{3}}}$ $\quad 3x^2-x-6$

$\qquad 7 \quad 14$

$3)3x+6 \quad 9$ $\quad \therefore Q=x+2$; and $R=9$

$x+2 \quad 9$

(2) $2)\frac{2x^3-3x+1}{\frac{x^2-1\frac{1}{2}x+\frac{1}{2}}{1\frac{1}{2}-\frac{1}{2}}}$ $\quad 2x^5-9x^4+5x^3+16x^2 \quad -16x+36$

$\qquad 3 \quad -1$

$\qquad -9+3$

$\qquad -7\frac{1}{2} \quad +2\frac{1}{2}$

$2)\; 2 \quad -6 \quad -5 \quad +11\frac{1}{2} \quad 17\frac{1}{2}-5\frac{3}{4}$

$1 \quad -3-2\frac{1}{2}+5\frac{3}{4} \quad 15/4+30\frac{1}{4}$

$\therefore Q=x^3-3x^2\times2\frac{1}{2}x+5\frac{3}{4}$; and $R=3\frac{3}{4}x+30\frac{1}{4}$

N.B. :—Note that R is constant in every case.

Arithmetical** Applications* (Miscellaneous***) :

We shall now take up a number of Arithmetical applications and get a clue as to the utility and jurisdiction of the ***Nikhilaṁ*** formula and why and where we have to apply the *Parivartya* ***Sūtra.***

(1) Divide *1234* by *112*

112	1		*234*
888			888
	1	**1**	**122**
	2	**1**	**122**
			888
	3		**010**
			888
	3+8		**898**
			—896
	11		**2**

But this is too cumbrous. The *Parivartya* formula will be more suitable. Thus—

112	*1*	*2*	3+ *4*
−1 −2		*−1*	*− 2*
			−1−2
	1	1	02

∴ Q=11 ; and R=2

This is ever so much simpler.

(2) Divide *1241* by **112.**

(i)

112	1		*241*
888			888
	1	**1**	**129**
			888
	2	**1**	**017**
			888
	3		**905**
	8		**—896**
	11		**9**

(ii) This too is too long. Therefore use ***Parāvartya***

112	**1**	**2**	**4**	**1**
−1−2		**—1**	**—2**	
			−1	*−2*
	11		**09**	

∴ Q=11 ; and **R=9**

(3) Divide 1234 by 160

(i) *Nikhilam* method **is** manifestly unsuitable. We **should** therefore use the ***Parāvartya*** formula.

(ii)

160	1 2	**3** 4
—6+0	–6	**0**
		240
	6	**274**
	1	– 160
	7	114

But this **is** a **case** where (***Vilokanenaiva***) i. e. by simple inspection or observation, **we** can put the **answer** down.

(4)

11203	**2 3**	**9 4 7 9**
—1—2—0—3	**—2**	**—4—0—6**
		—1—2—0—3
	21	**4 2 1 6**

(5)

112	**1 3 0**	**4 5**
—1—2	**—1—2**	**—4**
	—2	**4+8**
	12 4	**53**
	=116	**/ 53**

In **all** these cases (where the digits in the divisor are small) the *Nikhilam* method is generally unsuitable : and the *Parivartya* one **is** always to be preferred.

(6) Divide *13466* by 1123

1123	13	*4 6 6*
—1—2—3	**—1**	**—2—3**
		—2—4—6
	12	**0—2+0**

Here, as the **Remainder** portion is a negative quantity, we should follow the device used *in* subtractions of larger numbers from smaller ones *(in* **coinage etc)**.

Rs.	as.	**ps.**		**£.**	**s.**	**d**
7	*6*	*3*		**7**	**5**	**3**
	9	*9*			*9*	**9**
6	*11*	*6*		**6**	*16*	*6*

In other words, take **1** over from the quotient **column** to the remainder column **i.e.** take 1123 over to the right side, subtract 20 therefrom and **say, Q=11** and **R=1103**

(7) Divide 13905 by 113 **(similar)**

113	1 3 9	0 6
—1—3	—1—3	
	—2	—6
		—4 —12
	1 2 4	**— 107**
	1 2 3	**06**

N.B. :—Always remember that just as one **Rupee=16 annas,** One **Pound=20** shillings and one **Dollar=100 cents** and so on, so one taken over from the quotient to the Remainder--column stands, in concrete value, for the Divisor.

(8)

1012	**11**	**1 1 1**
0—1—2	**0**	**—1—2**
	11	**—1—1+2**
		+1012
	10	**991**

(9)

1133	**1 2**	**3 4 9**	
—1—3—3	**—1**	**—3—3**	
		—1—3—3	
	11	**—1—2+6**	**=—114**
		1133	
	10	1019	

(10) Divide 13999 by 112

112	1 3 9	9 9	
—1—2	—1—2		
	—2	**—4**	
		—5—10	
	1 2 5	**+111**	**∴Q=124 and R=111**

(11)

1132	11	3	2	9	Also by *Vilokanam* (i.e. mere observation).
—1—3—2	—1	—3	—2		
		0	0	0	
	10	**0**	**0**	**9**	

(12)

82	**1**	**0**	**3**	Also by *Vilokanam* (i.e. by mere observation)
1—2+2		2—2		
2—2				
	1	21		

(13) (i)

819	2	3	4	1	
$1\bar{8}1$		**2**	16	2	This is by the ***Nikhilam*** method
	2	**703**			

But 18 **can** be counted as **10+8** or as 20—2. **So,** put 181 down as **2—2+1.** We can thus ovoid multiplication by big digits **i.e. by** more than five.

(ii)

819	2	3	4	**1**
181		4		**—4+2**
2—2+1	**2**		703	

(14) Divide 39999 by 9819 or (ii) (by Vinculum and *Parāvartya*)

(i)

9819	3	9	9 9	9
$0\overline{18}1$		0	**324**	3
	3	1	0542	
			0181	
	4		0723	

(ii)

9819	3	9	9 9	9
$10\overline{22}1$		0	**6—6**	3
0+2—2+1	3	1	0642	
			02—2+1	
	3		0723	

(16) Divide 1111 by 839.

(i)

839	1	111
$\mathbf{1\bar{6}1}$		**161**
	1	272

or (ii) But **839=$\mathbf{1\bar{2}4\bar{1}}$**; and

161=$\mathbf{2\bar{4}1}$

∴

839	1	1	1	1
$\mathbf{1\bar{2}4\bar{1}}$		**2**	$\mathbf{\bar{4}}$	**1**
2—4—1	**1**	**3**	**3**	**2**
	=1	272		

(iii)

839	1	111
$\mathbf{1\bar{6}1}$		$\mathbf{2\bar{4}1}$
$\mathbf{2\bar{4}1}$	**1**	$\mathbf{3\bar{3}2}$
	1	272

(16) **818**
$\bar{1}82$

	5	0	1	2
		6	40	10
	6		922	
	=6		104	

(by simple subtraction of the Divisor as in the case of 16 annas, 20 shillings, 100 cents etc.)

(17) **988**
$\overline{012}$

13	**0**	**4**	**5**
0	**1**	**2**	
	0	**3**	**6**
13	**2**	**0**	**1**

(18) **858**
$\overline{142}$

7	**1**	**1**	**1**
	7	**28**	**14**
7	**1**	**105**	
		142	
8		**247**	

(19) **(i)** 828 43 9 9 **9**
$\bar{1}72$

43 9 9 **9**	
4 2 8 8	
7 4 9 1 4	
47 5	**0 8 3**
	5 3 5 1 0
52	**9 4 3**
	—828
53	**115**

or **(ii)** **828**
172
2—3+2

43 9	9 9
8—12	8
	22—33+22
51	**19—16+31**
51	**1 771**
	—2—32
52	**943**
	—828
53	**115**

(20) Divide 1771 by 828

(i) 828
$\bar{1}7\bar{2}$

1	771
	172
1	**943**
	—828
2	**115**

(ii) 828
172
2—3+2

1	7	7	1
	2—3+2		
1	**9**	**4**	**3**
	—8	**2**	**8**
2	**1**	**1**	**5**

(21) Divide 2671 by 828

(i) **828**
$\bar{1}72$

2	6	7	1
	2	**14**	**4**
2	1	015	
		172	
3		187	

or **(ii)** 828
172
2—3+2

2	6	7	1	
	4—6+4			
2	1	0	1	5
		2—3$2		
3		1	8	7

or (iii) Subtract 828 straight off (in both cases) from 1015.

(22) Divide 39893 by 829

(i)

829	39	8	9	3	0 r
$\overline{171}$	**3**	**21**	**3**		
		12	84	12	
	42	5	0 7 5		
			5 **3 5 5**		
	47		9 3 0		
			–8 2 9		
	48		1 0 1		

(ii)

829	**39**	**8**	**9**	**3**
171	**6–9+3**			
2–3+1		**30–45+15**		
	452	5	8	8
		4–6–2		
	47	9	3	0
		–8	2	9
	48	1	0	1

(23) Divide **21011** by **799**

799	21	0	1	1
$\overline{2}01$	4	0	**2**	
		10	0	5
	25 1		036	
			201	
	26		237	

(24) Divide 13046 by 988

988	13	0	4	5
012	**0**	**1**	**2**	
		0	3	6
	13	2	0	1

(25) Divide 21999 by 8819

(i)

8819	2	1	**9**	9	9
1181		**2+4—4+2**			
$\overline{1}$+2—2+1	**$\overline{2}$**	4	3	6	**$\overline{1}$**

or (ii)

8819	2	1	9	9	9
1118		**2**	**2**	**16**	**2**
	2	4	3	6	1

(26) Divide 1356 by 182 Even this is too cumbrous. *Ānurūpya* and *Parāvartya* will be more suitable.

(i)

182	13	5	6
–8–2	–8	**–2**	
		40	10
	5 4	4	6
		–32	–8
		–28	–2
	9	**+364**	
	–2		
	7		82

(ii)

182	13	6	6
2) 2—2+2	**1—1**		
1—1+1		4–4	
1–1	**2)14**	**8**	**2**
	7	8	2

N.B. R is constant.

(27)

882	3	1	**2**	8
118		**3+6—6**		
1+2—2	3	4	8	2

(28) (i)

882 | 4 0 0 9
1—1—2+2 | 4+8—8
1+2—2 | 4 4 8 1

(ii)

882 | 4 0 0 9
118 | 4+8—8
1+2—2 | 4 4 8 1

(29) Divide 4009 by 882

(i)

882 | 4 0 0 9
1—1—2+2 | 4+8—8
1+2—2 | 4 4 8 1

(ii)

882 | 4 0 0 9
118 | 4+8—8
1+2—2 | 4 4 8 1

Note:—In both these methods, the working is exactly the same.

(30)

2) 224 | 2 6 9 9
112 | —2—4
—1—2 | —4—8
2)24 11
12 11

R is constant

(31)

2) 223 | 1 6 9 9
111½ | —1—1½
—1—1½ | —5 —7½
2)15 2½+1½
7½ 2½ 6½
7 111½
2 138

(32)

2) 222 | 1 2 3 4
111 | —1 —1
—1—1 | —1—1
2)11 1 3
5 111
5 124

(33) Divide 7685 by 672

672 | 7 6 8 5
328 | 21+21 —14
3+3—2 | 7 2 9 8 1
6 +6 —4
9 1 6 3 7
3 +3 —2
10 9 6 5
—6 7 2
11 2 9 3

This work can be **curtailed—or** at least rendered a bit easier—by the ***Ānurūpyeṇa*** **Sūtra.** We can take 168 (which is one-fourth of 672) *or* 84 (which is one-eighth of **it**) or, better

still, 112 (which is one-sixth thereof); and work it out with that Divisor and finally divide the quotient *proportionately.*

The division (with 112 as Divisor) works out as follows:

```
∵ 672=6×112          7 6      8 5
∴  112                 —7      —14
  —1—2              ──────────────
                               1+2
                    ──────────────
                    7—1    —5+7
                    ──────────────
                 6) =69    —5+7
                    ──────────────
                      11½—5+7
                    =11    336—50+7=293
                    ──────────────
```

It will thus be seen that, in all such cases, a fairly easy method is for us to take the nearest multiple (or sub-multiple) to a power of 10 as our temporary divisor, use the *Nikhilaṁ* or the *Parāvartya* process and then multiply (or divide) the Quotient *proportionately.* A few more examples are given below, in illustration hereof:—

(1) Divide 1400 by 199

```
(i)    199        1 4 0 0
    2) 20—1         0  ½
       1+0—½           0+2
        0+½    ──────────────
              2)14    ½+2
               ──────────────
                 7    ½+2=7/7
               ──────────────
```

OR (ii) Since 5×199=995

```
  995        1    4    0    0
∴ 005   ────────────────0───0───5
             1    4    0    5
           ×5   —3    9    8
          ─────────────────────
             7    7
          ─────────────────────
```

(2) Divide 1699 by 223.

```
∵ 4×223=892
∴     892             1    6    9    9
   1—1—1+2                 1+1—2
    1+1—2          ─────────────────────
                      1    8    0    7
                     ×4  —6    6    9
                   ─────────────────────
                      4    1    3    8
                     —3
                   ─────────────────────
                      7    1    3    8
                   ─────────────────────
```

(3) Divide 1334 by 439

```
∵ 2×439=878
∴ 878      1    3    3    4
  122           1    2    2
         ─────────────────────
           1    4    5    6
          ×2  —4    3    9
         ─────────────────────
           3   17
         ─────────────────────
```

(4) Divide 1234 by 511.

$\because 2 \times 511 = 1022$

$\therefore \frac{1022}{0-2-2}$

1	2	3	4
	0	—2	—2
1	2	1	2
x 2			
2	2	1	2

(5) Divide 1177 by 516

$\because 2 \times 516 = 1032$

$\therefore \frac{1032}{0-3-2}$

1	1	7	7
	0	—3	—2
1	1	4	5
X2			
2	1	4	5

Note:—The Remainder is constant in all the cases.

CHAPTER VI

ARGUMENTAL DIVISION

(By simple argument per the ***Ūrdhva Tiryak Sūtra***)

In addition to the ***Nikhilaṁ*** method and the ***Parāvartya*** method (which are of use only in certain special cases) there is a third method of division which is one of simple argumentation (based on the '***Ūrdhva*** Tiryak' ***Sūtra*** and practically amounts to a converse thereof).

The following examples will explain and illustrate it :—

(1) Suppose we have to divide (x^2+2x+1) by $(x+1)$, we make a chart, as in the case of an ordinary multiplication (by the '***Ūrdhva Tiryak***' process) and jot down the dividend and the divisor. Then the argumentation is as follows :—

$$\begin{array}{r} x \;\; +1 \\ x \;\; +1 \\ \hline x^2+2x+1 \\ \hline \end{array}$$

(i) x^S and x being the first terms of the dividend and the divisor (or the product and the multiplier respectively), the first term of the quotient (or the multiplicand) ***must*** be x.

(ii) As for the coefficient of x in the product, it must come up as the sum of the cross-wise-multiplication-products of these. We have already got x by the cross-multiplication of the x in the upper row and the 1 in the lower row; but the coefficient of x in the product is 2. The other x must therefore be the product of the x in the lower row and the absolute term in the upper row. ∴ The latter is 1. And thus the Quotient is $x+1$.

(2) Divide $(12x^2-8x-32)$ by $(x-2)$.

$$\frac{12x^2-8x-32}{x-2} = 12x+16$$

(i) $12x^S$ divided by x gives us $12x$.

(ii) The twelve multiplied by —2 gives us —24 ; but the actual coefficient of x in the product (or the dividend) is —8 $\therefore$ We must get the remaining **16x** by multiplying the x of the divisor by 16. $\therefore$ The absolute term in the Divisor **must** be 16 $\therefore$ $Q=12x+16$. And as $-2\times16=-32$, $\therefore$ $R=0$.

$Q=12x-16$

(3) Divide (x^3+7x^2+6x+5) by $(x-2)$

(i) x^3 divided by x gives us x^2 which is therefore the first term of the quotient.

$$\frac{x^3+7x^2+6x+5}{x-2}$$

$\therefore$ $Q=x^2+9x+24$

(ii) $x^2\times-2=-2x^2$; but we have $7x^2$ in the Dividend. This means that we have to get $9x^2$ more. This must result from the multiplication of x by **9x.** Hence the second term of the divisor must be **9x.**

$$\frac{x^3+7x^2+6x+5}{x-2} \quad \therefore Q=x^2+9x+\ldots$$

(iii) As for the third term, we already have $-2\times9x=-18x$. But we have **6x** in the dividend. We must therefore get an additional **24x.** This can only come in by the multiplication of x by 24. $\therefore$ This is the third term of the quotient. $\therefore$ $Q=x^2+9x+24$

(iv) Now this last term of the quotient multiplied by —2 gives us **—48.** But the absolute term in the dividend is 5. We have therefore to get an additional **53** from somewhere. But there is no further term left in the Dividend. This means that the 53 will remain as the Remainder. $\therefore$ $Q=x^2+9x+24$; and $R=53$

Note :—All the work explained in detail above can be easily performed by means of the *'Parāvartya' Sūtra* (as already explained in the *'Pardvartya'* chapter, in connection with *Mental* division by Binomial divisors).

The procedure is very simple; and the following examples will throw further light thereon and give the necessary practice to the student :—

(1) $\dfrac{x^3+7x^2+9x+11}{x-2}$ ∴ $Q=x^2+9x+27$; and $R=65$

(2) $\dfrac{x^3-x^2-7x+3}{x-3}$ ∴ $Q=x^2+2x-1$; and R=O

(3) $\dfrac{x^4-3x^3+7x^2+5x+7}{x-4}$ ∴ $Q=x^3+{}^2+11x+49$; & $R=203$

(4) $\dfrac{-4x^3-7x^2+9x-12}{2x-4}$ ∴ $Q=-2x^2+\frac{1}{2}x+5\frac{1}{2}$; and $R=10$

(5) $\dfrac{3x2-x-5}{3x-7}$ ∵ $Q=x+2$; and R=9

(6) $\dfrac{16x^2+8x+1}{4x+1}$ ∴ $Q=4x+1$; and $R=0$

(7) $\dfrac{x^4-4x^2+12x-9}{x^2-2x+3}$ N.B. :—Put zero coefficients for absent powers.

∴ $Q=x^a+2x-3$; and R=O

(8) $\dfrac{x^3+2x^2+3x+5}{x^2-x-1}$ ∵ $Q=x+3$; and $R=7x+8$

(9) $\dfrac{x^4+4x^3+6x^2+4x+1}{x^2+2x+1}$ ∴ $Q=x^2+2x+1$; and $R=0$

(10) $\dfrac{x^4+2x^3+3x^2+2x+1}{x^2+x+1}$ ∴ $Q=x^2+x+1$; and $R=0$

(11) $\dfrac{x^4-x^3+x^2+5x+5}{x^2-x-1}$ ∴ $Q=x^2+4$; and $R=9x+9$

(12) $\dfrac{6x^4+13x^S+39x^2+37x+45}{x^2-2x-9}$ ∴ $Q=6x^2+25x+143$; and $R=548x+1332$

(13) $\dfrac{12x^4-3x^3-3x-12}{x^2+1}$ ∵ $Q=12x^2-3x-12$; and R=O

(14) $12x^4+48x^3+8kx^2 7+79x+42$ .. $Q=4x^2+7x+6$; & $R=0$

(15) $\dfrac{x^4-4x^2+12x-9}{x^2+2x-3}$ ∴ $Q=x^2-2x+3$ and $R=0$

(16) $\dfrac{12x^4-3x^3-3x-12}{12x^2-3x-12}=x^2+1$; and R=O

(17) $\dfrac{12x^4+41x^3+81x^2+79x+42}{4x^2+7x+6}=3x^2+5x+7$; and R=0

(18) $\dfrac{x^4-4x^2+12x-9}{x^2-2x+3}=x^2+2x-3$ and R=0

9 $\dfrac{2x^3+9x^2+18x+20}{\quad}=x^2+2x+4$; and R=O

(20) $\dfrac{2x^3+9x^2+18x+20}{x^2+2x+4}=x+5$; and R=O

(21) $\dfrac{6x^4+13x^3+39x^2+37x+45}{3x^a+2x+9}=2x^2+3x+5$; and R=O

(22) $\dfrac{6x^4+13x^3+39x^2+37x+45}{2x^2+3x+5}=3x^2+2x+9$ and R=0

(23) $\dfrac{16x^4+36x^2+81}{4x^2+6x+9}=4x^2-6x+9$; and R=O

(24) $\dfrac{16x^4+36x^2+81}{4x^2-6x+9}=4x^2+6x+9$; and R=O

(25) $\dfrac{16x^4+36x^2+16x+9}{4x^2+2x+1}=4x^2-2x+9$; and R=O

(26) $\dfrac{16x^4+36x^2+16x+9}{4x^2-2x+9}=4x^2+2x+1$; and R=O

(27) $\dfrac{x^5+x^3-7x^2+9}{x^3-2x^2+2x-3}=x^2+2x+3$; and $R=-2x^2+18$

(28) $\dfrac{16x^4+36x^2+6x+86}{4x^2+6x+9}=4x^2-6x+9$; & $R=6x+5$

(29) $\dfrac{-2x^5-7x^4+2x^3+18x^2-3x-8}{x^3-2x^2+1}=-2x^2-11x-20$;
and $R=-20x^2+8x+12$

(30) $\dfrac{x^4+3x^3-16x^2+3x+1}{x^2+6x+1}=x^2-3x+1$; and R=O

(31) $\dfrac{x^4+3x^3-16x^2+3x+1}{x^2-3x+1}=x^2+6x+1$; and R=O

(32) $\dfrac{2x^5-9x^4+5x^3+16x^2-16x+36}{x^2-3x+1}=x^3-3x^2-2\frac{1}{2}x+5\frac{3}{4}$;
and $R=3\frac{3}{4}x+30\frac{1}{4}$

(33) $$\frac{21x^6+7x^5+15x^4+29x^3+x^2+15x+3}{7x^8+5x+1}=3x^3+x^2+3$$

(34) $$\frac{21x^6+7x^5+15x^4+29x^3+x^2+15x+3}{3x^S+x^2+3}=7x^8+5x+1$$

(35)

$$\frac{7x^{10}+26x^9+53x^8+56x^7+43x^6+40x^5+41x^4+38x^3+19x^2+8x+5}{x^5+3x^4+5x^3+3x^2+x+1}$$
$$=7x^5+5x^4+3x^3+x^2+3x+5$$

(36)

$$\frac{\text{(Same dividend as above)}}{7x^5+5x^4+3x^3+x^2+3x+5}$$
$$=x^5+3x^4+5x^3+3x^2+x+1$$

LINKING NOTE

RECAPITULATION & CONCLUSION

OF

(Elementary) DIVISION SECTION

In these three chapters (IV, V and VI) relating to Division, we have dealt with a large number and variety of instructive examples and wc now feel justified in postulating the following conclusions :—

(1) The three methods expounded and explained are, no doubt, free from the big handicap which the current system labours under, namely, (i) the multiplication, of large numbers (the Divisors) by "trial digits" of the quotient at every step (with the chance of the product being found too big for the Dividend and so on), (ii) the subtraction of large numbers from large numbers, (iii) the length, cumbrousness, clumsiness etc, of the whole procedure, (iv) the consequent liability of the student to get disgusted with and sick of it all, (v) the resultant greater risk of errors being committed and so on ;

(2) And yet, although comparatively superior to the process now in vogue everywhere, yet, they too suffer, in some cases, from these disadvantages. At any rate, they do not, in such cases, conform to the Vedic system's Ideal of "Short and Sweet" ;

(3) And, besides, all the three of them are suitable only for some special and particular type (or types) of cases ; and none of them is suitable for general application to all cases :—

(i) The *'Nikhilam'* method is generally unsnitable for Algebraic divisions ; and almost invariably, the *'Parā-*vartya' process suits them better ;

(ii) and, even as regards Arithmetical computations, the *'Nikhilaṁ'* method is serviceable only when the Divisor-digits are large numbers (i.e., 6, 7, 8 or 9) and not at all helpful when the divisor digits are small ones (i.e. 1, 2, 3. 4 and 5) ; and it is only the

'Parāvartya' method that can be applied in the latter kind of cases!

(iii) Even when a convenient multiple (or sub-multiple) is made use of, even then there is room for a choice having to be made—by the pupil—as to whether the *'Nikhilam'* method or the 'Parcivartya' one should be preferred;

(iv) and there is no exception-less criterion by which the student tan be enabled to make the requisite final choice between the two alternative methods;

(v) and, as, for the third method (*i.e.* by the reversed *'Ūrdhva—Tiryak' Sūtra*), the Algebraic utility thereof is plain enough; but it is difficult in respect of Arithmetical calculations to say when, where and why it should be resorted to (as against the other two methods).

All these considerations (arising from our detailed-comparative-study of a large number of examples) add up, in effect, to the simple conclusion that none of these methods can be of general utility in all cases, that the selection of the most suitable method in each particular case may (owing to want of uniformity) be confusing to the student and that this element of uncertainty is bound to cause confusion. And the question therefore naturally—my, unavoidably arises as to whether the Vedic Sūtras can give us a General Formula applicable to all cases.

And the answer is:—Yes, most certainly YES! There *is* a splendid and beautiful and very easy method which conforms with the Vedic ideal of ideal simplicity all-round and whirh in fact gives us what we have been describing as "Vedic one line-mental answers"!

This astounding method we shall, however, expound in a later chapter under the caption "Straight-Division"—which is one of the Crowning Beauties of the Vedic mathematics Sūtras. (Chapter XXVII. q.v.).

———

CHAPTER VII

I. FACTORISATION (of Simple Quadratics)

Factorisation comes in naturally at this point, as a form of what we have called "Reversed multiplication" and as a particular application of division. There is a lot of strikingly good material in the Vedic Siitras on this subject too, which is new to the modern mathematical world but which comes in at a very early stage in our Vedic 'Mathematics.

We do not, however, propose to go into a detailed and exhaustive exposition of the subject but shall content ourselves with a few simple sample examples which will serve to throw light thereon and especially on the Sūtraic technique by which a Sūtra consisting of only one or two simple words, makes comprehensive provision for explaining and elucidating a procedure whereby a 80-called "difficult" mathematical problem (which, in the other system puzzles the students' brains) ceases to do so any longer, nay, is actually laughed at by them as being worth rejoicing over and not worrying over!

For instance, let us take the question of factorisation of a quadratic expression into its component binomial factors. When the coefficient of x^2 is 1, it is easy enough, even according to the current system wherein you are asked to think out and find two numbers whose algebraic total is the middle coefficient and whose product is the absolute term. For example, let the quadratic expression in question be $x^2+7x+10$, we mentally do the multiplication of the two factors $(x+2)$ and $(x+5)$ whore product is $x^2+7x+10$; nd (by a mental process of reverting thereof), we think of 2 and 5 whose sum is 7 and whose product is 10: and we thus factorise $(x^2+7x+10)$ into $(x+2)$ and $(x+5$. And the actual working out thereof is as follows:—

$$\begin{array}{r} x+2 \\ x+5 \\ \hline x^2+7x+10 \\ \hline \end{array}$$

$$\begin{aligned} &x^2+7x+10 \\ &=x^2+2x+5x+10 \\ &=x(x+2)+5\ (x+2) \\ &=(x+2)\ (x+5) \end{aligned}$$

The procedure is, no doubt, mathematically correct; but the process is needlessly long and cumbrous. However, as the mental process actually employed is as explained above, there is no great harm done.

In respect, however, of Quadratic expressions whose first coefficient is not unity (e.g. $2x^2+5x+2$), the students do not follow the mental process in question but helplessly depend on the 4-step method shown above and work it out as follows: –

$$2x^2+5x+2$$
$$=2x^2+4x+x+2$$
$$=2x(x+2)+1\ (x+2)$$
$$=(x+2)\ (2x+1)$$

As the pupils are never taught to apply the mental process which can give us this result immediately, it means a real injury. The Vedic system, however, prevents this kind of harm, with the aid of two small sub-Siitras which say (i) आनुरूप्येण (*Ānurūpyeṇa*) and (ii) आद्यमाद्येनान्त्यमन्त्येन (*Ādyamādyenāntya-mantyena*) and which mean 'proportionately' and 'the first by the first and the last by the last'.

The former has been explained already (in connection with the use of multiples and sub-multiples, in multiplication and division); but, alongside of the latter sub-Siitra, it acquires a new and beautiful double application and significance and works out as follows:—

(i) Split the middle coefficient into two such parts that the ratio of the first coefficient to that first part is the same as the ratio of that second part to the last coefficient. Thus, in the qundratic $2x^a+5x+2$, the middle term (5) is split into two such parts (4 and 1) that the ratio of the first coefficient, to the first part of the middle coefficient (i.e. 2 : 4) and the ratio of the second part to the last coefficient (i.e. 1 : 2) are the same. Now, this ratio (i.e. $x+2$) is one factor.

And the second factor is obtained by dividing the first coefficientof the Quadratic by the first coefficient of the factor

already found and the last coefficient of the Quadratic by the last coefficient of that factor. In other words the second Binomial factor is obtained thus: $\frac{2x^2}{x}+\frac{2}{2}=2x+1$.

Thus we say: $2x^2+5x+2=(x+2)\ (2x+1)$.

Note:—The middle coefficient (which we split-up above into (4+1) may also be split up into (1+4), that the ratio in that case is (2x+1) and that the other Binomial factor (according to the above-explained method) is (x+2). Thus, the change of SEQUENCE (in the splitting up of the middle term) makes no difference to the factors themselves!

This sub-Sūtra has actually been used already (in the chapters on division); and it will be coming up again and again, later on (i.e. in Co-ordinate Geometry etc., in connection with straight lines, Hyperbolas, Conjugate Hyperbolas, Asymptotes etc.)

But, just now, we make use of it in connection with the factorisation of Quadratics into their Binomial factors. The following additional examples will be found useful:—

(1)	$2x^2+5x-3$	=	$(x+3)\ (2x-1)$
(2)	$2x^2+7x+5$	=	$(x+1)\ (2x+5)$
(3)	$2x^2+9x+10$	=	$(x+2)\ (2x+5)$
(4)	$2x^2-5x-3$	=	$(x-3)\ (2x+1)$
(5)	$3x^2+x-14$	=	$(x-2)\ (3x+7)$
(6)	$3x^2+13x-30$	=	$(x+6)\ (3x-5)$
(7)	$3x^2-7x+2$	=	$(x-2)\ (3x-1)$
(8)	$4x^2+12x+5$	=	$(2x+1)\ (2x+5)$
(9)	$6x^2+11x+3$	=	$(2x+3)\ (3x+1)$
(10)	$6x^2+11x-10$	=	$(2x+5)\ (3x-2)$
(11)	$6x^2+13x+6$	=	$(2x+3)\ (3x+2)$
(12)	$6x^2-13x-19$	=	$(x+1)\ (6x-19)$
(13)	$6x^2+37x+6$	=	$(x+6)\ (6x+1)$
(14)	$7x^2-6x-1$	=	$(x-1)\ (7x+1)$

(15) $8x^2-22x+5 = (2x-5)(4x-1)$
(16) $9x^2-15x+4 = (3x-1)(3x-4)$
(17) $12x^2+13x-4 = (3x+4)(4x-1)$
(18) $12x^2-23xy+10y^2 = (3x-2y)(4x-5y)$
(19) $15x^2-14xy-8y^2 = (3x-4y)(5x+2y)$

An additional sub-Sūtra is of immense ultility in this context, for the purpose of verifying the correctness of our answers in multiplications, divisions and factorisations. It reads : गुणितसमुच्चयः समुच्चयगुणितः and means :—

> "The product of the sum of the *coefficients* in the factors is equal to the sum *of* the *coefficients* in the *product*".

In symbols, we may put this principle down thus :—

S_c of the product=Product of the S_c (in the factors).

For example, $(x+7)(x+9)=(x^2+16x+63)$;

and we observe that $(1+7)(1+9)=1+16+63=80$

Similarly, in the case of Cubics, Bi-quadratics etc., the same rule holds good. For example:

$(x+1)(x+2)(x+3)=x^3+6x^2+11x+6$;

and we observe that $2\times3\times4=1+6+11+6=24$.

Thus, if and when some factors are known, this rule helps us to fill in the gaps.

It will be found useful in the factorisation of cubics, biquadratics etc., and will be adverted to (in that context and in some other such contexts) later on.

CHAPTER VIII

FACTORISATION (of "Harder" Quadratics)

There is a class of Quadratic expressions known as Homogeneous Expressions of the second degree, wherein several letters (x, y, z etc.) figure and which are generally fought shy of by students (and teachers too) as being too "difficult" but which can be very easily tackled by means of the *Ādyamādyena* Sūtra (just explained) and another sub-Siitra which consists of only one (compound) word, which reads लोपनस्थापनाभ्यां and means :—"by (alternate) Elimination and Retention"

Suppose we have to factorise the Homogeneous quadratic **($2x^a+6y^a+3z^2+7xy+11yz+7zx$).** This is obviously a case in which the ratios of the coefficients of the various powers of the various letters are difficult to find out ; and the reluctance of students (and even of teachers) to go into a troublesome thing like this, is quite understandable.

The 'Lopam—Sthipam' sub-Sūtra, however, removes the whole difficulty and makes the factorisation of a Quadratic of this type as easy and simple as that of the ordinary quadratic (already explained). The procedure is as follows :—

Suppose we have to factorise the following long Quadratic :

$$2x^2+6y^2+3z^2+7xy+11yz+7zx.$$

(i) We first eliminate z (by putting $z=0$) and retain only x and y and factorise the resulting ordinary quadratic (in x and y) (with the *Ādyam Sūtra*) ;

(ii) We then similarly eliminate y and retain only x and a and factorise the simple quadratic (in x and z) ;

(iii) with these two sets of factors before us, we fill in the gaps caused by our own deliberate elimination of z and y respectively. And that gives us the real factors of the given long expression. The procedure is an argumentative one and is as follows :—

If $z=0$, then E (the given expression)$=2x^a+7xy+6y^2$
$=(x+2y)(2x+3y)$.

Similarly, if $y=0$, then $E=2x^2+7xy+3z^2=(x+3z)(2x+z)$

$\therefore$ Filling in the gaps which we ourselves had created by leaving out z and y, we say : $E=(x+2y+3z)(2x+3y+z)$

The following additional examples will be found useful :—

(1) $3x^2+y^2-2z^2-4xy+yz-zx$
$E=(x-y)(3x-y)$ and also $(x-z)(3x+2z)$
$\therefore E=(x-y-z)(3x-y+2z)$

(2) $3x^a+xy-2y^2+19xz+28z^2+9xy-30w^2-yz+19wy+46zw$.

By eliminating two letters at a time, we get :
$E=(x+y)(3x-2y)$, $(3x+4z)(x+7z)$ and also $(x-2y)(3x+15w)$
$\therefore E=(x+y+4z-2w)(3x-2y+4z+15w)$

(3) $2x^2+2y^2+5xy+2x-5y-12=(x+3)(2x-4)$ and also $(2y+3)(y-4)$
$\therefore E=(x+2y+3)(2x+y-4)$

(4) $3x^2-8xy+4y^2+4y-3=(x-1)(3x+3)$ and also $(2y-1)(2y+3)$
$\therefore E=(x+2y-1)(3x+2y-3)$

(5) $6x^2-8y^2-6z^2+2xy+16yz-5xz$
$=(2x-2y)(3x+4y)$ and also $(2x+3z)(3x-2z)$
$\therefore E=(2x-2y+3z)(3x+4y-2z)$

Note :—We could have eliminated x also and retained only y and z and factorised the resultant simple quadratic. That would not, however, have given us any additional material but' would have only confirmed and verified the answer we had already obtained. Thus, when 3 letters (x, y and z) are there, only two eliminations will generally suffice. The following exceptions to this rule should be noted :—

(1) $x^2+xy-2y^2+2xz-5yz-3z^2$
$=(x-y)(x+2y)$ and $(x-z)(x+3z)$

But x is to be found in all the terms; and there is no means for deciding the proper combinations.

In this case, therefore, x too may be eliminated; and y and z retained. By so doing, we have :—

$E = -2y^2-5yz-3z^2=(-y-z)(2y+3z)$

$\therefore$ $E=(x-y-z)(x+2y+3z)$.

OR, avoid the x^2 (which gives the same co-efficient) and take only y^2 or z^2. And then, the confusion caused by the oneness of the coefficient (in all the 4 factors) is avoided; and we get,

$E=(x-y-z)(x+2y+3z)$ (as before).

(2) $x^2+2y^2+3xy+2xz+3yz+z^2$.

(i) By eliminating z, y and x one after another, we have $E=(x+y+z)(x+2y+z)$

OR (ii) By y or z both times, we get the same answer.

(3) $x^2+3y^2+2z^2+4xy+3xz+7yz$

Both the methods yield the same result:

$E=(x+y+2z)(x+3y+z)$

(4) $3x^2+7xy+2y^2+11xz+7yz+6z^2+14x+8y+14z+8$.

Here too, we can eliminate two letters at a time and thus keep only one letter and the independent term, each time.

Thus, $E= 3x^2+14x+8=(x+4)(3x+2)$;

$2y^2+8y+8=(2y+4)(y+2)$; and also

$6z^2+14z+8=(3z+4)(2z+2)$

$\therefore$ $E=(x+2y+3z+4)(3x+y+2z+2)$

Note:—This "*Lopana—Sthāpana*" method (of alternate elimination and retention) will be found highly useful, later on in H.C.F., in Solid Geometry and in Co-ordinate Geometry of the straight line, the Hyperbola, the Conjugate Hyperbola, the Asymptotes etc.

CHAPTER IX

FACTORISATION OF CUBICS ETC.

(By Simple Argumentation e. t. c.)

We have already seen how, when a polynomial is divided by a Binomial, a Trinomial etc., the remainder can be found by means of the Remainder Theorem and how both the Quotient and the Remainder can be easily found by one or other method of division explained already.

From this it follows that, if, in this process, the remainder is found to be zero, it means that the given dividend is divisible by the given divisor, i.e. the divisor is a factor of the Dividend.

And this means that, if, by some such method, we are able to find out a certain factor of a given expression, the remaining factor (or the product of all the remaining factors) can be obtained by simple division of the expression in question by the factor already found out by some method of division. (In this context, the student need hardly be reminded that, in all *Algebraic* divisions, the *'Pardvartya'* method is always to be preferred to the *'Nikhilam'* method).

Applying this principle to the case of a cubic, we may say that, if, by the Remainder Theorem or otherwise, we know one Binomial factor of a cubic, simple division by that factor will suffice to enable us to find out the Quadratic (which is the product of the remaining two binomial factors). And these two can be obtained by the *'Ādyamādyena'* method of factorisation already explained.

A simpler and easier device for performing this operation will be to write down the first and the last terms by the *'Ādyamādyena'* method and the middle term with the aid of the *Guṇita-Samuccaya* rule (i.e. the principle—already explained with regard to the S_c of the product being the same as the Product of the S_c of the factors).

Let us take a concrete example and see how this method can be made use of. Suppose we have to factorise $x^3+6x^2+11x+6$ and that, by some method, we know $(x+1)$ to be a factor. We first use the *Ādyamādyena* formula and thus mechanically put down x^2 and 6 as the first and the last coefficients in the quotient (i.e. the product of the remaining two binomial factors). But we know already that the S_c of the given expression is 24 ; and, as the S_c of $(x-1)=2$ we therefore know that the S_c of the quotient must be 12. And as the first and last digits thereof are already known to be 1 and 6, their total is 7. And therefore the middle term must be $12-7=5$. So, the quotient is x^2+5x+6.

This is a very simple and easy but absolutely certain and effective process.

The student will remember that the ordinary rule for divisibility of a dividend by a divisor (as has been explained already in the section dealing with the "Remainder—Theorem") is as follows :—

If $E=DQ+R$, if $D=x-p$ and if $x=p$, then $E=R$.

COROLLARIES

(i) So, if, in the dividend, we substitute 1 for x, the result will be that, as all the powers of 1 are unity itself, the dividend will now consist of the sum of all the coefficients.

Thus, if D is $x-1$, $R=a+b+c+d+$(where a, b, c, d etc., are the successive coefficients) ; and then, if $a+b+c$ etc., $=0$, it will mean that as $R=0$, E is divisible by D. In other words, $x-1$ is a factor.

(ii) If, however, $D=x+1$ and if we substitute -1 for x in E, then, inasmuch as the odd powers of -1 will all be -1 and the even powers thereof will all be 1, therefore it will follow that, in this case, $R=a-b+c-d$ etc.

So, if $R=0$ i.e. if $a-b+c-d$ etc., $=0$, i.e. if $a-b+c-d$ etc., $=0$, i.e. $a+c+\ldots=b+d+\ldots$.

i.e. if the sum of the coefficients of the odd powers of x and the sum of the coefficients of the even powers be equal, then $x+1$ will be a factor.

The following few illustrations will elucidate the actual application of the principle mainly by what may be called the Argumentation method, based on the simple multiplication-formula to the effect that—

$(x+a)(x+b)(x+c)=x^3+x^2(a+b+c)+x(ab+ac+bc)$ $+abc$, as follows :—

(1) Factorise $x^3+6x^2+11x+6$.

(i) Here, $S_c=24$; and t_1 (the last term) is 6 whose factors are 1, 2, 3 or 1, 1, 6. But their total should be 6 (the coefficient of x^a). So we must reject the 1, 1, 6 group and accept the 1, 2, 3 group. And, testing for the third coefficient, we find $ab+bc+ca=11$
$\therefore E=(x+1)(x+2)(x+3)$.

or (ii) S_o (the sum of the coefficients of the odd powers) $=1+11=12$; and S_e (the sum of the coefficients of the even powers)$=6+6=12$. And as $S_o=S_e$ $\therefore x+1$ is a factor.
$\therefore$ Dividing E by that factor, we first use the '*Ādyamādyena*' Sūtra and put down I and 6 as the first and the last coefficients. $\therefore$ The middle coefficient is $12-(1+6)=5$ $\therefore$ The $Q=x^2+5x+6$ which (by *Ādyamādyena*)$=(x+2)(x+3)$.
Thus $E=(x+1)(x+2)(x+3)$.

(2) Factorise $x^3-6x^2+11x-6$

Here $S_c=0$ $\therefore x-1$ is a factor. But as $\div$ is an indefinite figure, we cannot use the *Guṇita-Samuccaya* method here for the middle term but must divide out (by mental '*Parāvartya*') and get the quotient as x^2-5x+6 which (by the *Ādyamādya*' rule) $=(x-2)(x-3)$ $\therefore E=(x-1)(x-2)(x-3)$.

or (ii) argue about -1, -2 and -3 having -6 as the total and -6 as the product; and test out and verify the 11. And therefore say, $E=(x-1)(x-2)(x-3)$.

(3) Factorise $x^3+12x^2+44x+48$.

(i) Here $S_c=105$ whose factors are 1, 3, 5, 7, 15, 21, 35 and 105. And t_l is 48 whose factors are, 1, 2, 3, 4, 6,8 12, 16, 24 and 48. $\therefore$ $x+1$ and $x-1$ are out of court. And the only possible factors are $x+2$, $x+4$ and $x+6$ (verify).

(ii) or, argue that $2+4+6=12$ and $2\times4\times6=48$; and test for and verify 44 $\therefore$ $E=(x+2)(x+4)(x+6)$

(4) Factorise $x^3-2x^2-23x+60$

(i) Here $S_c=36$ (with factors 1, 2, 3, 4, 6, 9, 12, 18 and 36; $t_l=60$ (which is $1\times2\times2\times3\times5$.)

$\therefore$ Possible factors are 1, 2, 3, 4, 5, 6, 10, *12*, 15, *20*, *30* and *60*. But the sum of the coefficients in each factor must be a factor of the total S_c. (i.e. 105). Therefore, all the italicized numbers go out, and so do $x-1$, $x+4$, $x+6$ and $x+10$.

Now, the only possible numbers here (which when added, total -2) are -3, -4 and 5. Now, test for and verify $x-3$

$\therefore$ $E=(x-3)(x^2+x-20)=(x-3)(x-4)(x+5)$.

(ii) or take the possibilities $x-10$, $x-5$, $x+5$, $x-4$, $x+3$, $x-3$, $x+2$ and $x-2$.

$$\text{If } x-2=\frac{0}{2}\quad \begin{array}{l} x^3-2x^2-23x+60 \\ \quad 2+\quad\quad 0-46 \\ \hline 1+0+23 \quad 14 \end{array} \qquad \therefore R=14$$

$\therefore$ $x-2$ is not a factor.

But if $x-3$, R=O $\therefore$ $x-3$ *is* a factor.

Then, argue as in the first method.

(6) Factorise x^3-2x^2-5x+6=Here $S_c=0$

(i) $\therefore$ $x-1$ is a factor; and the other part (by Division) is x^2-x-6 which $=(x+2)(x-3)$

$\therefore$ $E=(x-1)(x+2)(x-3)$

(ii) $t_l=6$ (whose factors are 1, 2 and 3). And the only combination which gives us the total -2, is -1, 2 and -3. Test and verify for -5. And put down the answer.

(6) Factorise $x^3+3x^2-17x-38$

Now $S_c=-51$ (with factors ±1, ±3, ±17 and ±51

And —38 has the factors ±1, ±2, ±19 and ±38.

$\therefore$ ±1, ±19 and ±38 are not possible.

And only ±2 is possible. And if $x=-2$, $R=0$

$\therefore$ x+2 is a factor $\therefore$ $E=(x+2)\ (x^2+x-19)$

which has no further factors.

(7) Factorise $x^3+8x^a+19x+12$.

(i) Here $S_c=8$; and $L_t=12$ $\therefore$ $1+3+4$ are the proper numbers. Now test for and verify 19.

$\therefore$ $E=(x+1)\ (x^2+7x+12)=(x+1)\ (x+3)\ (x+4)$

OR (ii) $\because$ $1+19=8+12$ $\therefore$ $x+1$ is a factor. Then the quotient is obtainable by the '*Ādyamādyena*' and '*Samuccaya*' Sūtras. And that again can be factorised with the aid of the former.

$\therefore$ $E=(x+1)\ (x+3)\ (x+4)$

(8) Factorise x^3-7x+6

(i) $\because$ $S_c=0$, $\therefore$ $x-1$ is a factor.

$\therefore$ By '*Parāvartya*' method of division (mental).

$E=(x-1)\ (x^2+x-6)=(x-1)\ (x-2)\ (x+3)$

OR (ii) (by a different kind of application of *Ādyamādyena*

$$x^3-7x+6=(x^3-1)-7x+7=(x-1)(x^2+x+1-7)$$
$$=(x-1)\ (x-2)\ (x+3)$$

Note : (1) This method is always applicable when x^2 is absent; and this means that the 3 independent terms together total zero.

(2) Note the note on this and other allied points (in the section relating to cubic equations) in a later chapter.

(3) Note that this method of factorisation by Argumentation is equally applicable to Biquadratics also.

(4) The relationship between the Binomial factors of a polynomial and its differentials (first, second and so on) is an interesting and intriguing subject which will be dealt with in a later chapter.

(5) The use of differentials for finding out repeated factors will also be dealt with later.

CHAPTER X

HIGHEST COMMON FACTOR

In the current system of mathematics, we have two methods which are used for finding the H.C.F. of two (or more) given expressions.

The first is by means of factorisation (which is not always easy); and the second is by a process of continuous division (like the method used in the G.C.M. chapter of Arithmetic). The latter is a mechanical process and can therefore be applied in all cases. But it is rather too mechanical and, consequently, **long** and cumbrous.

The Vedic method provides a third method which is applicable to all cases and is, at the same time, free from this disadvantage.

It is, mainly, an application of the 'Lopanu-Sthipanu' Sūtra, the '*Saṅkalana-Vyavakalan*' process and the '*Ādyamādya*' rule. The procedure adopted is one of alternate destruction of the highest and the lowest powers (by a suitable multiplication of the coefficients and the addition or subtraction of the multiples). A concrete example will elucidate the process :---

(I) Suppose we have to find the H.C.F. of (x^2+7x+6) & (x^2-5x-6)

(i) $x^2+7x+6=(x+1)(x+6)$; and $x^2-5x-6=$ $=(x+1)(x-6)$ $\therefore$ The H.C.F. is $(x+1)$.

(ii) The 2nd method (the G.C.M. one) is well-known and nee not be put down here.

(iii) The third process of '*Lopana-Sthāpana*' (i.e. of Elimination and Retention, or Alternate destruction of the highest and the lowest powers) is explained below :

Let E_1 and E, be the two expressions. Then, for destroying the highest power, we should subtract E_2 from E_1; and for

destroying the lowest one, we should add the two. The chart is as follows:

$$\left.\begin{array}{r} x^2+7x+6 \\ x^2-5x-6 \end{array}\right\} \text{Subtraction} \qquad \left.\begin{array}{r} x^2-5x-6 \\ x^2+7x+6 \end{array}\right\} \text{Addition}$$

$$12)\ \underline{12x+12} \qquad\qquad 2x)\ \underline{2x^2+2x}$$

$$x+1 \qquad\qquad\qquad x+1$$

We then remove the common factor (if any) from each; and we find $(x+1)$ staring us in the face.

$\therefore$ $x-1$ is the H.C.F.

The Algebraical *principle* or Proof hereof is as follows :—

Let P and Q be the two expressions; H their H.C.F. and A and B the quotients (after their division by the H.C.P.)

$\therefore \frac{P}{H}=A$; and $\frac{Q}{H}=B$ $\therefore P=HA$ and $Q=HB$

$\therefore P\pm Q=H\ (A\pm B)$; and $MP\pm NQ=H(MA\pm NB)$

$\therefore$ The H.C.F. of P and Q is also the H.C.P. of $P\pm Q$, $2P\pm Q$, $P\pm 2Q$ and $MP\pm NQ$

All that we have therefore to do is to select our M and N in such a way that the highest and the lowest powers are removed and the H.C.F. appears and *shows* itself before us.

A few more illustrative examples may be seen below :—

(I) (i) $x^3-3x^2-4x+12=(x+2)(x-2)(x-3)$;

and $x^3-7x^2+16x-12=(x-2)^2(x-3)$

$\therefore$ the H.C.F. is $(x-2)(x-3)=x^2-5x+6$

But the factorisation of the two expressions will be required.

or (ii) The G.C.M. method.

or (iii) The '*Lopana-Sthāpana*' method :—

$$\begin{array}{r} x^3-3x^2-4x+12 \\ -(x^3-7x^2+16x-12) \\ \hline 4)\ \underline{4x^2-20x+24} \\ x^2-5x+6 \\ \hline \end{array} \qquad \begin{array}{r} x^3-7x^2+16x-12 \\ +(x^3-3x^2-4x+12) \\ \hline 2x)\ \underline{2x^3-10x^2+12x} \\ x^2-5x+6 \\ \hline \end{array}$$

$\therefore$ The H.C.F. is (x^2-5x+6)

(2) (i) $4x^3+13x^2+19x+4=(4x+1)(x^2+3x+4)$;
and $2x^3+5x^2+5x-4=(2x+1)(x^2+3x+4)$

$\therefore$ The H.C.F. is x^2+3x+4

But the factorisation of the two cubics will he cumbrous.

or (ii) The G.C.M. method.

or (iii) The Vedic method :—

$$\begin{array}{r} 4x^3+13x^2+19x+4 \\ -(4x^3+10x^2+10x-8) \\ \hline 3)\ 3x^2+9x+12 \\ \hline x^2+3x+4 \end{array} \qquad \begin{array}{r} 2x^3+5x^2+5x-4 \\ +(4x^3+13x^2+19x+4) \\ \hline 6x)\ 6x^3+18x^2+24x \\ \hline x^8+3x+4 \end{array}$$

$\therefore$ The H.C.F. is (x^2+3x+4)

(3) (i) $x^4+x^3-5x^2-3x+2=(x+1)(x-2)(x^2+2x-1)$;
and $x^4-3x^3+x^2+3x-2=(x+1)(x-2)(x-1)^2$

$\therefore$ The H.C.P. is x^2--x-2

But this factorisation of the two biquadratics is bound to be a (comparatively) laborious process.

(ii) The cumbrous G.C.M. method.

(iii) The Vedic method :—

$$\begin{array}{r} x^4+x^3-5x^2-3x+2 \\ -(x^4-3x^3+x^2+3x-2) \\ \hline 2)\ 4x^3-6x^2-6x+4 \\ \hline 2x^3-3x^2-3x+2 \\ -(2x^3-2x^2-4x) \\ \hline -1(-x^2+x+2) \\ \hline x^2-x-2 \end{array} \qquad \begin{array}{r} x^4-3x^3+x^2+3x-2 \\ x^4+x^3-5x^2-3x+2 \\ \hline 2x^2)\ 2x^4-2x^3-4x^2 \\ \hline x^2-x-2 \end{array}$$

(*N.B.*—multiply this by 2x & take it over to the left for subtraction).

$\therefore$ The **H.C.F** is x^2-x-2

(4) (i) **The** Vedic method :—

$$\begin{array}{r} 6x^4-7x^3-5x^2+14x+7 \\ -(6x^4-10x^3\qquad\quad +14x) \\ \hline 3x^3-5x^2\qquad\quad +7 \\ \hline \end{array} \qquad 3x^3-5x^2+7$$

(*N.B.*—multiply this by 2x & subtract from L.H.S.)

(ii) The factorisation of the big Biquadratic will be "harder'?.

(iii) The G.C.M. method is, in this case, easy. But how should one know this beforehand and start tnonkeying or experimenting with it ?

(5) (i) *The Vedic method :—*

$$\begin{array}{ll} 6x^4-11x^3+16x^2-22x+8 & 6x^4-11x^3-8x^2+22x-8 \\ -(6x^4-11x^3-8x^2+22x-8) & +(6x^4-11x^3+16x^2-22x+8) \\ \hline 4)\ 24x^2-44x+16 & 2x^2)\ 12x^4-22x^3+8x^2 \\ \hline 6x^2-11x+4 & 6x^2-11x+4 \end{array}$$

$\therefore$ The H.C.F. is $6x^2-11x+4$

(ii) $6x^4-11x^3+16x^2-22x+8=(2x-1)(3x-4)(x^2+2)$; &
$6x^4-11x^3-8x^2+22x-8=(2x-1)(3x-4)(x^2-2)$
$\therefore$ The H.C.F. is $(2x-1)(3x-4)=6x^2-11x+4$

(iii) The cumbersome G.C.M. method.

(6) (i) $2x^3+x^2-9=(2x-3)(x^2+2x+3)$;
and $x^4+2x^2+9=(x^2+2x+3)(x^2-2x+3)$
$\therefore$ The H.C.F. is x^2+2x+3

But the factorisation-work (especially of the former expression) will be a toughish job.

(ii) The G.C.M. method will be cumbrous (as usual).

(iii) The Vedic method—

$$\begin{array}{ll} 2x^8+x^a-9 & 2x^4+4x^2+18 \\ x^4+2x^2+9 & 2x^4+x^3-9x \\ \hline x^a)\ x^4+2x^3+3x^2 & x^3-4x^2-9x-18 \\ \hline x^a+2x+3 & x^3+2x^2+3x \\ & \hline -6)\ -6x^2-12x-18 \\ & \hline x^a+2x+3 \end{array}$$

N.B.—As this hab no further factors, it must be in the R.H.S. Multiply it by x and take it over to the right for subtraction.

$\therefore$ The H.C.P. is x^2+2x+3

(7) (i) $4x^4+11x^3+27x^2+17x+5$ and $3x^4+7x^3+18x^2+7x+5$

(ii) $\therefore\ 12x^4+33x^S+81x^2+51x+15$

$$\begin{array}{rl}
 & 12x^4+33x^S+81x^2+51x+15 \\
 & \underline{12x^4+28x^3+72x^2+28x+20} \\
 & 5x^3+9x^2+23x-5 \\
 & \underline{5x^3+20x^2+45x+50} \\
-11\ ) & \underline{11x^2-22x-55} \\
 & \underline{x^2+2x+5}
\end{array}$$

$$\begin{array}{rl}
 & 4x^4+11x^3+27x^2+17x+5 \\
 & \underline{3x^4+7x^3+18x^2+7x+5} \\
x\ ) & \underline{x^4+4x^3+9x^2+10x} \\
 & x^3+4x^2+9x+10 \\
 & \underline{10x^3+18x^2+46x-10} \\
11x\ ) & \underline{11x^3+22x^2+55x} \\
 & \underline{x^a+2x+5}
\end{array}$$

(ii) The G.C.M. method will be cumbrous (as usual).

(i) $4x^4+11x^3+27x^2+17x+B=(x^2+2x+5)\ (4x^2+3x+1)$ &
$3x^4+7x^3+18x^2+7x+5=(x^2+2x+5)\ (3x^2+x+1)$

But the factorisation of the two big biquadratics into two further factorless quadratics each, will entail greater waste of time and energy.

So, the position may be analysed thus :—

(i) The G.C.M. method is mechanical and reliable but too cumbrous ;

(ii) The Factorisation method is more intellectual hut harder to work out and therefore less dependable ;

(iii) The Vedic method is free from all these defects and is not only intellectual but also simple, easy and reliable. And the beauty of it is that the H.C.F. places itself before our eyes and seems to stare us in the face !

———

CHAPTER XI

SIMPLE EQUATIONS (FIRST PRINCIPLES)

As regards the solution of equations of various types, the Vedic sub-Sūtras give us some First Principles which are theoretically not unknown to the western world but are not (in actual practice) utilised as basic and fundamental first principles of a practically Axiomatic character (in mathematical computations).

In order to solve such equations, the students do not generally use these basic sub-Sūtras as such but (almost invariably) go through the whole tedious work of practically proving the formula in question instead of taking it for granted and applying it ! Just as if on every occasion when the expression $a^3+b^3+c^3-3abc$ comes up ; one should not take it for granted that its factors are $(a+b+c)$ and $(a^a+b^8+c^a-ab-bc-ca)$ but should go through the long process of multiplying these two, showing the product and then applying it to the case on hand, similarly for Pythagoras Theorem etc. !

The Vedic method gives us these sub-formulae in a condensed form (like *Parāvartya* etc.,) and enables us to perform the necessary operation by mere application thereof. The underlying principle behind all of them is परावर्त्य योजयेत् (*Parāvartya Yojayet*) which means : "Transpose and adjust" The applications, however, are numerous and splendidly useful. A few examples of this kind are cited hereunder, as illustrations thereof :—

(1) $2x+7=x+9 \therefore 2x-x=9-7 \therefore x=2$. The student has to perform hundreds of such transposttion-operations in the course of his work ; but he should by practice obtain such familiarity with and such master over it as to assimilate and assume the general form a that if $ax+b=cx+d$, $x=\frac{d-b}{a-C}$ and apply it by mental arithmetic automatically

to the particular case actually before him and say :—

$2x+7=x+9 \therefore x=\frac{9-7}{2-1}=\frac{2}{1}=2$; and

the whole process should be a short, and simple mental process.

Second General Type

(2) The above is the commonest kind (of transpositions). The second common type is one in which each side (the L.H.S. and the R.H.S.) contains two Binomial factors.

In general terms, let $(x+a)(x+b)=(x+c)(x+d)$. The usual method is to work out the two multiplications and do the transpositions and say :—

$$(\sim+a)(x+b)=(\sim+e)(x+d)$$
$$\therefore x^2+ax+hx+ab=x^Z+cx+dx+cd$$
$$\therefore ax+bx-cx-dx=cd-ab$$
$$\therefore x(a+b-c-d)=cd-ab$$
$$\therefore x=\frac{cd-ab}{a+b-c-d}$$

It must be possible for the student, by practice, to assimilate and assume the whole of this operation and say immediately : $x=\frac{cd-ab}{a+b-c-d}$

As examples, the following may be taken :—

(1) $(x+1)(x+2)=(x-3)(x-4) \therefore x=\frac{12-2}{1+2+3+4}=\frac{10}{10}=$,

(2) $(x-6)(x+7)=(x+3)(x-11) \therefore x=\frac{-33+42}{-6+7-3+11}=\frac{9}{9}=1$

(3) $(x-2)(x-5)=(x-1)(x-4) \therefore x=\frac{4-10}{-2-5+1+4}=\frac{-6}{-2}=3$

(4) $(x-7)(x-9)=(x-3)(x-22) \therefore x=\frac{66-63}{-7-9+3+22}=\frac{3}{9}=\frac{1}{3}$

(5) $(x+7)(x+9)=(x+3)(x+22) \therefore x=\frac{66-63}{7+9-3-22}=\frac{3}{-9}=\frac{1}{-3}$

(6) $(x+7)(x+9)=(x-8)(x-11) \therefore x=\frac{88-63}{7+9+8+11}=\frac{25}{35}=\frac{5}{7}$

(7) $(x+7)(x+9)=(x+3)(x+21) \therefore x=\frac{63-63}{7+9-3-21}=\frac{0}{-8}=0$

This gives rise to a general corollary to the effect that, if $cd-ab=0$ i.e. if $cd=ab$ i.e. if the product of the absolute terms be the same on both the sides, the numerator becomes zero; and $\therefore$ $x=0$.

Third General Type

The third type is one whirh may be put into the general form :— $\frac{ax+b}{cx+d}=\frac{p}{q}$; and, after doing all the cross-multiplication and transposition etc.. lie get : $x=\frac{bq-dp}{aq-cp}$ The student should (by practice) be able to assimilate and assume this also and do it all ***mentally*** as a single operation.

Note : -The only rule to remember (for facilitating this process) is that all the terms involving x should be conserved on to the left side and that all the independent terms should be gathered together on the right side and that every transposition for this purpose must invariably produce a change of sign (i.e. from + to — and conversely; and from X into ÷ and conversely).

Fourth General Type

The fourth type is of the form :— $\frac{m}{x+a}+\frac{n}{x+b}=0$

After all the L.C.M's, the cross—multiplications and the transpositions etc., are over, we get. $x=\frac{-mb-na}{m+n}$ This is simple enough and easy enough for the student to assimilate; and it should be assimilated and readily applied mentally to any case before us.

In fact, the application of this process may, in clue course, by means of practice, be extended so as to cover cases involving a larger number of terms. For instance,

$$\frac{m}{x+a}+\frac{n}{x+b}+\frac{p}{x+c}=0$$

$$\therefore \frac{m(x+b)(x+c)+n(x+c)(x+a)+p(x+a)(x+b)}{(x+a)(x+b)(x+c)}=0$$

$\therefore\ m[x^2+x(b+c)+bc]+n[x^2+x(c+a)+ca]+$
$+p[x^2+x(a+b)+ab]=0$

$\therefore\ x^2(m+n+p)+x[m(b+c)+n(c+a)+p(a+b)]+$
$(mbc+nca+pab)=0$

If $m+n+p=0$, then

$$x=\frac{-mbc-nca-pab}{m(b+c)+n(c+a)+p(a+b)}$$

Rut if $m+n+p\pm0$, then it will be a Quadratic equation and will have to be solved as such (as explained in a later chapter).

And this method can be extended to any number of terms (on the same lines as explained above).

LINKING NOTE

Special Types of Equations

The above types may be described as General types. But there are, as in the cave of multiplications, divisions etc,, particular types which possess certain specific characteristics of a SPECIAL character which can be more easily tackled (than the ordinary ones) with the aid of certain very short SPECIAL processes (practically what one may describe as *mental* one-linc methods).

As already explained in a previous context, all that the student has to do is to look for certain characteristics. spot them out, identify the particular type and apply the formula which is applicable thereto.

These SPECIAL types of equations, we now go on to, in the next few chapters.

CHAPTER XII

SIMPLE EQUATIONS (*by Sūtra Śūnyam* etc.,)

We begih this section with an exposition of several special types of equations which can be solved practically at sight-with the aid of a beautiful special Sūtra which reads: ?@ साम्यसमुच्चये (*'Śūnyam Sāmyasamuccaye'*) and which, in cryptic language (which renders it applicable to a large number of different eases) merely says: ' when the Samuccaya is the same, that *Samuccaya* is zero" i.e. it should be equated to zero.

'Samuccaya' is a technical term which has several meanings (under different contexts); and we shall explain them, one by one:

FIRST MEANING AND APPLICATION

'Xamuccaya' first means a term which occurs as a common factor in all the terms concerned.

Thus $12x+3x=4x+5x \therefore 12x+3x-4x-5x=0$
$\therefore 6x=0 \therefore x=0$

All these detailed steps are unnecessary; and, in fact, *no one* works it out in this way. The mere fact that x occurs as a common factor in all the terms on both sides [or on tbe L.H.S. (with zero on the R.H.S.)] is sufficient for the inference that x is zero; and no intermediate step is necessary for arriving at this conclusion. This is practically axiomatic.

And this is applicable not only to x or other such "unknown quantity" but to every such case. Thus, if $9(x+1)=7(x+1)$, we need not say: $9(x+1)=7(x+1)$
$\therefore 9x+9=7x+7 \therefore 9x-7x=7-9 \ldots 2x=-2 \therefore x=-1$.
On the contrary, we can straightaway say: $9(x+1)=7(x+1)$
$\therefore x+1=0 \therefore x=-1$

SECOND MEANING AND APPLICATION

The word *'Samuccaya'* has, as its second meaning, the product of the independent terms. Thus, $(x+7)(x+9)=(x+3)(x+21) \therefore$ Here $7 \times 9=3 \times 21$. Therefore $x=0$.

This is also practically axiomatic, has been dealt with in a previous section (of this very subject of equations) and need not be gone into again.

Third Meaning and Application

'*Samuccaya*' thirdly means the sum of the Denominators of two fractions having the same (numerical) numerator. Thus

$$\frac{1}{2x-1}+\frac{1}{3x-1}=0 \quad \therefore 5x-2=0$$

This is axiomatic too and needs no elaboration.

Fourth Meaning and Application

Fourthly, '*Samuccaya*' means combination (or total). In this sense, it is used in several different contexts; and they are explained below :—

(i) If the sum of the numerators and the sum of the denominators be the same, then that sum—zero. Thus :

$$\frac{2x+9}{2x+7}=\frac{2x+7}{2x+9}$$

$$\therefore (2x+9)(2x+9)=(2x+7)(2x+7)$$

$$\therefore 4x^2+36x+81=4x^2+28x+49$$

$$\therefore 8x=-32$$

$$\therefore x=-4$$

This is the current method. But the "*Sūnyam Sāmya-Samuccaye*" formula tells us that, inasmuch as $N_1+N_2=4x+16$ and D_1+D_2 is also $4x+16$ $\therefore 4x+16=0$ $\therefore x=-4$. In fact, as soon as this characteristic is noted and the type recognised, the student can at once mentally say : $x=-4$.

Note: —If in the algebraical total, there be a numerical factor that should be removed. Thus:

$$\frac{3x+4}{6x+7}=\frac{x+1}{2x+3}$$

Here $N_1+N_2=4x+5$; and $D_1+D_2=8x+10$. Removing the numerical factor, we have $4x+5$ on both sides here too.

$$\therefore 4x+5=0 \quad \therefore x=-5/4$$

No laborious cross-multiplications of N_1 by D_2 and N_2 by D_1 and transpositions etc., are necessary (in the Vedic method).

At sight, we can at once say $4x+5=0$ and be done with it.

Fifth Meaning and Application (for Quadratics)

With the same meaning (i.e. total) of the word समुच्चय '*Samuccaya*', there is a fifth kind of application possible of this Sūtra. And this has to do with Quadratic equations. None need, however, go into a panic over this. It is as simple and as easy as the fourth application : and even little children can understand and readiiy apply this Sūtra in this context, as explained below.

In the two instances given above, it will be observed that the cross-multiplications of the coefficients of x gives us the same coefficient for x^2. In the first case, we had $4x^a$ on both sides; and in the second example, it was $6x^2$ on both sides. The two cancelling out, we had simple equations to deal with.

But there are other cases where the coefficients of x^2 are not the same on the two sides; and this means that we have a quadratic equation before us.

But it does not matter. For, the same Sūtra applies (although in a different direction) here too and gives us also the second root of the quadratic equation. The only difference is that inasmuch as Algebraic: '*Samuccaya*' includes subtraction too, we therefore now take into account, not only the sum of N_1 and N_2 and the sum of D_1 and D_2 but also the differences between the numerator and the denominator on each side; and, if they be equal, we at once equate that difference to Zero.

Let us take a concrete example and suppose we have *to* solve the equation $\frac{3x+4}{6x+7}=\frac{5x+6}{2x+3}$

(i) We note that $N_1+N_2=8x+10$ and D_1+D_2 is also $8x+10$; we therefore use the method described in the fourth application given above and equate $8x+10$ to zero and say: $x=-5/4$.

(ii) But mental cross-multiplication reveals that the x^2 coefficients (on the L.H.S. and the R.H.S.) are 6 and 30 respectively and not the same. So, we decide that it is a quadratic equation ; and we observe that $N_1 \sim D_1 = 3x+3$ and that $N_2 \sim D_2$ is also $3x+3$. And so, according to the present application of the same Sūtra, we at once say : $3x+3=0 \quad \therefore x=-1$.

Thus the two roots arc $-5/4$ and -1 ; and we have solved a quadratic equation at mere sight (without the usual paraphernalia of cross-multiplication. transposition etc.). We shall revert to this at a later stage (when dealing with quadratic equations themselves, as such).

Sixth Meaning and Application

With the same sense 'total' of the word '*Samuccaya*' but in a different application, we have the same Siitra coming straight to our rescue, in the solution of what the various text-books everywhere describe as "Harder Equations", and deal with in a very late chapter 'thereof under that caption. In fact, the label "Harder" has stuck to this type of equations to such an extent that they devote a separate section thereto and the Matriculation examiners everywhere would almost seem to have made it an invariable rule of practice to include one question of this type in their examination-papers !

Now, suppose the equation before us is :—

$$\frac{1}{x-7}+\frac{1}{x-9}=\frac{1}{x-6}+\frac{1}{x-10}$$

In all the text-books, we are told to transpose two of the terms (so that each side may have a plus term and a minus term), take the L.C.M. of the denominators, cross-multiply, equate the denominators, expand them, transpose and so on and so forth. And, after 10 or more steps of working, they tell you that 8 is the answer.

The Vedic Siitra, however, tells us that, if (other elements being equal), the sum-total of the denominators on the L.H.S. and the total on the R.H.S. be the same, then that total is zero !

In this instance, as D_1+D_2 and D_3+D_4 both total $2x-16$, $\therefore 2x-16=0 \therefore x=8$! And that is all there is to it ! A few more instances may be noted :—

(1) $\frac{1}{x+7}+\frac{1}{x+9}=\frac{1}{x+6}+\frac{1}{x+10} \quad \therefore x=-8$

(2) $\frac{1}{x-7}+\frac{1}{x+9}=\frac{1}{x+11}+\frac{1}{x-9} \quad \therefore x=-1$

(3) $\frac{1}{x-8}+\frac{1}{x-9}=\frac{1}{x-5}+\frac{1}{x-12} \quad \therefore x=8\frac{1}{2}$

Disguised Specimens

The above were plain, simple cases which could be readily recognised as belonging to the type under consideration. There, however, are several cases which really belong to this type but come under various kinds of disguises (thin, thick or ultra-thick)! But, however thick the disguise may be, there are simple devices by which we can penetrate and see through the disguises and apply the *'Śūnya Samuccaye'* formula :—

Thin Disguises

(1) $\frac{1}{x-8}-\frac{1}{x-5}=\frac{1}{x-12}-\frac{1}{x-9}$

Here, we should transpose the *minuses*, so that all the 4 terms are ***plus* ones** :—

$$\frac{1}{x-8}+\frac{1}{x-9}=\frac{1}{x-12}+\frac{1}{x-5} \quad \therefore x=8\frac{1}{2}$$

The transposition-process here is very easy and can be done mentally (in less than the proverbial trice).

(2) $\frac{1}{x+1}-\frac{1}{x+3}=\frac{1}{x+2}-\frac{1}{x+4} \quad \therefore x=-2\frac{1}{2}$

(3) $\frac{1}{x+1}-\frac{1}{x-3}=\frac{1}{x-4}-\frac{1}{x-8} \quad \therefore x=3\frac{1}{2}$

(4) $\frac{1}{(x-b)\quad x-b-d}=\frac{1}{x-c+d}-\frac{1}{x-c} \quad \therefore x=\frac{1}{2}(b+c)$

(5) $\frac{1}{x+b}-\frac{1}{x+b+d}=\frac{1}{x+c-d}-\frac{1}{x+c} \quad \therefore x=-\frac{1}{2}(b+c)$

Note:—If the last two examples (with so many literal coefficients involved) were to be done according to the current svstem, the labour entailed (over the L.C.M.'s, the multiplications etc.,) would have been terrific ; arid the time taken would have been proportionate too ! Rut, by this (Vedic) method, the equation is solved *ut sight* !

Medium Disguises

The above were cases of thin disguises, where mere transposition was sufficient for enabling us to penetrate them. We now turn to cases of disguises of medium thickness :—

(1) $\frac{x-2}{x-3}+\frac{x-3}{x-4}=\frac{x-1}{x-2}+\frac{x-4}{x-5}$

By dividing the Numerators out by the Denominators, we have :

$$1+\frac{1}{x-3}+1+\frac{1}{x-4}=1+\frac{1}{x-2}+1+\frac{1}{x-5}$$

Cancelling out the two ones from both sides, we have the Equation before us in its undisguised shape and can at once say, $\therefore x=3\frac{1}{2}$.

Now, this process of division can be mentally performed very easily, thus :—

(i) $\frac{x}{x}+\frac{x}{x}=\frac{x}{x}+\frac{x}{x}$ $\because$ $(1+1=1+1)$

(ii) Applying the *Parāvartya* method (mentally) and transferring the independent term of the denominator (with its sign changed) to the Numerator, we get 1 as the result in each of the 4 cases.

With the help of these two TESTS, we know that "the other elements are the same" ; and, as $D_1+D_2=D_3+D_4$, we therefore identify the case before us as coming completely within the jurisdiction of the "*Sūnyam Samuccaye*" formula

$\therefore$ $2x-7=0$ $\therefore x=3\frac{1}{2}$

(2) $\frac{x}{x-2}+\frac{x-9}{x-7}=\frac{x+1}{x-1}+\frac{x-8}{x-6}$

Here, $\frac{1}{1}+\frac{1}{1}=\frac{1}{1}+\frac{1}{1}$;

Secondly, by *Parāvartya,*

$$\frac{2}{x-2}-\frac{2}{x-7}=\frac{2}{x-1}-\frac{2}{X-6}$$

We transpose the *minus* terms and find that all the TESTS have been satisfactorily passed. (All this argumentation can of course, be done mentally).

So, we say: $2x-8=0 \quad \therefore x=4$

(3) $\frac{2x-3}{x-2}+\frac{3x-20}{x-7}=\frac{x-3}{x-4}+\frac{4x-19}{x-5}$

Here $\frac{2}{1}+\frac{3}{1}=\frac{1}{1}+\frac{4}{1}$; the Numerators all become 1; and $D_1+D_2=D_3+D_4=2x-9=0 \quad \therefore x=4\frac{1}{2}$

(4) $\frac{3x-8}{x-3}+\frac{4x-35}{x-9}=\frac{2x-9}{x-5}+\frac{5x-34}{x-7}$

Here, $\frac{3}{1}+\frac{4}{1}=\frac{2}{1}+\frac{5}{1}$; and the other 2 tests are all right too. $\therefore x=6$

(5) $\frac{3x-13}{x-4}+\frac{4x-41}{x-10}=\frac{2x-13}{x-6}+\frac{5x-41}{x-8}$

All the TESTS are found satisfactorily passed. $\therefore 2x-14=0 \quad \therefore x=7$

(6) $\frac{4x+21}{x+5}+\frac{5x-69}{x-14}=\frac{3x-5}{x-2}+\frac{6x-41}{x-7}$

All the TESTS are all right $\therefore 2x-9=0$

(7) $\frac{x^2+3x+3}{x+2}+\frac{x^2-15}{x-4}=\frac{x^2+7x+11}{x+5}+\frac{x^2-4x-20}{x-7}$

Either by simple division or by simple factorisation (both of them, mental). we note :—

(i) $(x+1)+(x+4)=(x+2)+(x+3)$

(ii) the numerators are all unity; and

(iii) $D_1+D_2=D_3+D_4=2x-2=0 \quad \therefore x=1$

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THICKER DISGUISES

(1) $\frac{2}{2x+3}+\frac{3}{3x+2}=\frac{1}{x+1}+\frac{6}{6x+7}$

(i) At first sight, this does ***not*** seem to be of the type which we have been dealing with in this section. But we note that the coefficient of x in the four denominators is not the same. So, by suitable multiplication of the numerator and the denominator in each term, we get **6** (the L.C.M. of the four coefficients) uniformly as the coefficient of x in all of them. Thus, we have :—

$$\frac{6}{6x+9}+\frac{6}{6x+4}=\frac{6}{6x+6}+\frac{6}{6x+7}$$

Now, we can readily recognise the type and say :–

$12x+13=0 \quad \therefore x=\frac{-13}{12}$

But we cannot gamble on the possible chance of its being of this type and go through all the laborious work of L.C.M., the necessary multiplications etc , (and perhaps find at the end of it all, we have drawn a blank) ! There must therefore be some valid and convincing test whereby we can satisfy ourselves beforehand on this point (and, if convinced, ***then*** *and then only* should we go through all the toil involved).

And that test is quite simple and easy :—

$\frac{2}{2}+\frac{3}{3}=\frac{1}{1}=\frac{6}{6}$. But even then, only the *possibility* or the *probability* (and not the certainty) of it follows therefrom.

(ii) A second kind of **TEST**—with guarantee of certainty—is available too. And this is by **CROSS-multiplication** of N_1 by D, and of N, by D_1 on the one hand and of N_3 by D_4 and of N_4 by D_3 on the other. (And this too can be done mentally).

Thus, in the case dealt with, we get from each side—the same $12x+13$ as the **total** $\therefore 12x+13=0$

$\therefore x=\frac{-13}{12}$

(2) $\frac{3}{3x+1}-\frac{6}{6x+1}=\frac{3}{3x+2}-\frac{2}{2x+1}$

(i) We transpose (mentally) and note :

$\frac{3}{3}+\frac{2}{2}=\frac{3}{3}+\frac{6}{6}$ So, we may try the L.C.M. method.

$\frac{6}{6x+2}+\frac{6}{6x+3}-\frac{6}{6xf1}+\frac{6}{6x+4}=0$

$\therefore$ $12x+5=0$ $\therefore$ $x=5\ \frac{-5}{12}$

(ii) Even here, after the preliminary testing of $\frac{3}{3}+\frac{2}{2}$ being equal to $\frac{3}{3}+\frac{6}{6}$, we may straight away CROSS-multiply and say : $\therefore$ $12x+5=0$ .. $x=-5/12$

(3) $\frac{3}{3x+1}+\frac{2}{2x-1}=\frac{3}{3x-2}+\frac{2}{2x+1}$

$\therefore$ By either of the two methods, u-c get $12x-1=0$

$\therefore$ $x=\frac{1}{12}$

(4) $\frac{1}{x+3}+\frac{3}{3x-1}=\frac{1}{x+5}+\frac{3}{3x-7}$

$\therefore$ By either method, $6x+8=0$ $\therefore$ $x=\frac{-4}{3}$

(5) $\frac{2x+11}{x+5}\ \frac{9x+9}{3x-4}\ \frac{4x+13}{x+3}\ \frac{15x-47}{3x-10}$

Hrre $\frac{2}{2}-\frac{9}{3}=\frac{4}{1}-\frac{15}{3}$ $\therefore$ YES.

By simple division, we pot this into proper shape, as follows :—

$\frac{1}{x+5}+\frac{3}{3x-10}=\frac{1}{x+3}+\frac{3}{3x-4}$

Here $\frac{1}{1}+\frac{3}{3}=\frac{1}{1}+\frac{3}{3}$ $\therefore$ YES.

$\therefore$ By either method, $6x+5=0$ $\therefore$ $x=\frac{-5}{6}$

(6) $\frac{5-6x}{3x-1}+\frac{2x+7}{x+3}=\frac{31-12x}{3x-7}+\frac{4x+21}{x+5}$

$\therefore$ $\frac{1}{x+3}+\frac{3}{3x-1}=\frac{1}{x+5}+\frac{3}{3x-7}$

$\therefore$ By either method, $6x+8=0$ $\therefore$ $x=\frac{-4}{3}$

FURTHER APPLICATIONS OF THE FORMULA

(1) In the case of a special type of seeming "cubics" :—

There is a certain type of equations which look like cubic equations hut, which (after wasting a huge lot of our time and energy) turn out to be simple equations of the first degree and which come within the range of the *"Śūnyam Samuccaye"* formula. Thus, for instance—

$$(x-3)^3+(x-9)^3=2(x-6)^3$$

The current system works this out at enormous length (by expanding all the three cubes, multiplying, transposing etc..) and finally gives us the answer $x=6$

The Vedic Sūtra now under discussion is, however, applicable to this kind of case too and says :—

$(x-3)+(x-9)=2x-12$. Taking away the numerical factor, we have $x-6$. And $x-6$ is the factor under the cube on R.H.S. $\therefore x-6=0 \quad \therefore x=6$

The Algebraical proof of it is as follows :

$$(x-2a)^3+(x-2b)^3=2(x-a-b)^3$$

$$\therefore x^3-6x^2b+12xa^2-8a^3+x^3-6x^2b+12xb^2-8b^3=$$

$$=2(x^3-3x^2a-3x^2b+3xa^2+3xb^2+6xab-a^3-3a^2b-3ab^2-b^3)$$

$$=2x^3-6x^2a-6x^2b+6xa^2+6x^2b+12xab-2a^3-6a^2b-6ab^2-2b^3$$

Cancelling out the common terms from both sides, we have :

$$12xa^2+12xb^2-8a^3-8b^3=6xa^2+6xb^2+12xab-2a^3-6a^2b-6ab^2-2b^3$$

$$6xa^2+6xb^2-12xab=6a^3-6a^2b-6ab^2+6b^3$$

$$\therefore 6x(a-b)^2=6(a+b)(a-b)^2$$

$$\therefore x=a+b$$

Obviously this particular oomhination was not thought, of and worked out by the mathematicians working under the current system. At any rate, it is not found listed in their books (under any known formula or as a conditional identity" and so on). The Vedic mathematicians, howevor, seem to have worked it all out and given us the benefit thereof (by the application of this formula to examples of this type).

We need hardly point out that the expansions, multiplications, additions, transpositions, factorisations in each particular case of this type must necessarily involve the expenditure of tremendous time and energy while the Vedic formula gives us the answer *at sight* !

Three more illustrations may be taken :—

(i) $(x-149)^3+(x-51)^3=2(x-100)^3$

The very prospect of the squaring. cubing etc., of these numbers must appal the student. But, by the present Sūtra we can at once say . $2x-200=0 \quad \therefore x=100$

(ii) $(x-249)^3+(x+247)^3=2(x-1)^3$

This is still more terrific. RIB~,with the aid of this Sūtra, we can at once say : $x=1$; and

(iii) $(x+a+b-c)^3+(x+b+c-a)^3=2(x+b)^3$

The literal coefficients make this still worse. But the Vedic one-line mental answer is : $x=-b$.

(2) *In the case of a special type of seeming "Biquadratics"* :

There is also similarly, a special type of *seemingly* "Biquadratic" equations which are really of the first degree and which the same Sūtra solves for us, at sight. Than. for example :

$$\frac{(x+3)^3}{(x+5)}=\frac{x+1}{x+7}$$

According to the current method, we cross-multiply and say . $(x+7)(x+3)^3=(x+1)(x+5)^3$

Expanding the two sides (with the aid of the usual formula

$$[(x+a)(x+b)(x+c)(x+d)=x^4+x^3(a+b+c+d)$$
$$+x^2(ab+ac+ad+bc+bd+cd)+x$$
$$\times(abc+abd+acd+bcd)+abcd)] \quad \text{(twice over)},$$

we will next say :—

$$x^4+16x^3+90x^2+216x+189=x^4+16x^3+9x^2+200x+125$$

Cancelling out the common terms and transposing, we then sny :

$\therefore 16x=-64 \quad \therefore x=-4$

According to the Vedic formula, howevr, we do not cross-multiply the binomial factors and so on but simply *observe* that N_1+D_1* and N_2+D_2 are both $2x+8$ and $\therefore 2x+8=0$ $\therefore x=-4$

The Algebraic proof hereof is as follows:—

$$\frac{(x+a+d)^3}{(x+a+2d)}=\frac{x+a}{x+a+3d}$$

$\therefore$ By the usual process of cross-multiplications,

$$(x+a+3d)^3(x+a+d)=(x+a)(x+a+2d)^3$$

$\therefore$ (By expansion of both sides)

$$x^4+x^3(4a+6d)+x^2(16a^2+18ad+12d^2)+x(4a^3+18a^2d+24ad^2+10d^3)+(a^4+6a^3d+12a^2d^2+12ad^3+d^4)=x^4+x^3(4x+6d)+x^2(16a^2+18ad+12d^2)+x(4a^3+18a^2d+4ad^2+8d^3)+\text{etc., etc.}$$

$\therefore$ (Cancelling common terms out), we have :—

$$x(10d^3)+10ad^3+3d^4=x(8d^3)+8ad^3.$$

$\therefore 2d^3x+2ad^3+3d^4=0$

$\therefore$ (Cutting d^3 out), we have $2x+2a+3d=0$ $\therefore x=-\frac{1}{2}(2a+3d)$

At this point, the student will note that N_1+D_1 (under the cubes) and N_2+D_2 are both $(2x+2a+3d)$. And this gives us the required clue to the particular characteristic which characterises this type of equations. i. e. that N_1+D_1 (under the cube) and N_2+D_2 must be the same; and, obviously, therefore, the '*Sūnyam Samuccaye*' Sūtra applies to this type. And, while the current system has evidently not tried, experienced and listed it, the Vedic seers had doubtless experimented on, observed and listed this particular combination also and listed it under the present Sūtra.

Note :—(1) The condition noted above (about the 4 Binomials) is interesting. The sum of the first + the second must be the same as the sum of the 3rd and 4th.

(2) The most obvious and readily understandable condition fulfilling this requirement is that the absolute terms in N,, N_1, D_1 and D_2 Binomials should be in Arithmetical Progression.

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*(within the cubes)

(3) This may also be postulated in this way, i.e. that the difference between the two Binomials on the R.H.S. must he equal to thrice the difference between those on the L.H.S. This, however, is only a coroilary—result arising from the A.P. relationship amidst the four Binomials (namely, that if N_2, N_1, D_1 and D_2 are in A.P. it is obvious that $D_2-N_2=3(D_1-N_1)$.

(4) In any case, the formula (in this special type) may be enunciated—in general terms—thus:—if N+D on both sides be the same, N+D should bc equated to zero :

Two more examples of this type may be taken :—

(1) $\frac{(x-5)^3}{(x-7)}=\frac{(x-3)}{(x-9)}$ $\therefore$ $2x-12=0$ $\therefore$ $x=6$

(2) $\frac{(x-a)^3}{(x+b)}=\frac{x-2a-b}{x+a+2b}$

Working all this out (which all the literal coefficients and with cross-multiplicatons, expansions, cancellations, transpositions etc., galore) would be a horrid task (for even the most laborious labourer). The Vedic formula, however, tells us that $(x-a)+(x-b)$ and $(x-2a-b)+(x+a+2b)$ both total up to $2x-a+b$ $\therefore$ $x=\frac{1}{2}(a-b)$

Note:—In all the above examples, it will be observed that the 4 binomials are not merely in Arithmetical Progression but are also so related that their cross totals are also the same.

Thus, in the first example worked out above, by Cross-multiplication, we have $(x+7)(x+3)^3=(x+1)(x+5)^3$; and the Cross-ADDITION of these factors gives us $4x+16$ as the total on both sides; and this tallies with the value $x=-4$ (obtained above).

In the second example :—

$$(x+a+3d)\ (x+a+d)^3=(x+a)(x+a+2d)^3.$$

And here too, Cross-ADDITION gives us $4x+4a+6d$ as the total on both sides. And this too gives us the same answer as before.

In the third example, we have :—

$$(x-9)(x-5)^3=(x-3)(x-7)^3.$$

And here too the Cross-ADDITION Process gives us $4x-24$ as the total on both sides. And we get the same answer as before.

In the fourth case, we have :—

$$(x+a+2b)(x-a)^3=(x-2a-b)(x+b)^3.$$

And cross-ADDITION again gives us the total $4x-2a+2b$ on both sides and, therefore, the same value of x as before.

The student should not, however, fall into the error of imagining that this is an *additional* TEST (or sufficient condition) for the application of the formula. This really comes in as a corollary-consequence of the A.P. relationship between the Binomial factors. But it is not a *sufficient* condition (by itself) for the applicability of the present formula. The Rule about N_1+D_1 and N_2+D_2 being the same, is the only condition sufficient for this purpose.

An instance in point is given below :—

$$\frac{(x+3)^3}{(x+5)}=\frac{x+2}{x+8}$$

Here, Cross-ADDITION gives $4x+17$ as the total on both sides, and the condition $D_2-N_2=3(D_1-N_1)$ is also satisfied (as $6=2\times3$). But $3+5+2+8$; and. as this essential condition is lacking, this particular equation does *not* come within the purview of this Sūtra.

On actual cross-multiplication and expansion etc., we find :—

$$x^4+17x^3+105x^2+275x+250=x^4+17x^3+99x^2+243x+216$$

$\therefore$ $6x^2+32x+34=0$, which is a Quadratic equation (with the two Irrational Roots $\frac{-16\pm\sqrt{52}}{6}$) and not a simple equation at all. of the type we are here dealing with.

And this is in conformity with the lack of the basic condition in question i.e. that N_1+D_1 and N_2+D_2 should be the same.

(3) ***In the! case of another special type of seeming "Biquadratics"***

There is also another special type of seeming "Biquadratic~"which are really simple equations of the first degree. which the *"Śūnyam Samuccaye"* Sūtra is applicable to and which we now go on to. (This section may. however, be held over for a later reading).

$$\frac{(x+1)\ (x+9)}{(x+2)\ (x+4)}=\frac{(x+6)\ (x+10)}{(x+5)\ (x+7)}$$

or $(x+1)(x+9)(x+5)(x+7)=(x+2)(x+4)(x+6)(x+10)$

We first note that cross-ADDITION gives us the same total $(4x+22)$ on both sides. This gives us the assurance that, on cross-multiplication, expansion etc. the x^4 and the x^3 coefficients will cancel out. Rut what about the x^2 coefficients ?

For them too to vanish, it is necessary that the sum of the products of the independent terms taken two at a time should be the same on both the sides. And this is the case when if $(x+a)(x+b)\ (x+c)\ (x+d)=(x+e)\ (x+f)\ (x+g)\ (x+h)$, we have not merely $a+b+c+d=e+f+g+h$ but also two other conditions Fulfilled :—

(i) that the sum of *any* 2 binomials on the one side is ual to the sum of *some* two binomials on the other :

and (ii) $ab+cd$ on the left$=ef+gh$ on the right.

In the example actually now before us. we find all these conditions fulfilled :—

(1) $(x+1)+(x+9)=(x+4)+(x+6)$; $(x+1)+(x+5)=(x+2)+(x+4)$: $(x+1)+(x+7)=(x+2)+(x+6)$; $(x+9)+(x+5)=(x+4)+(x+10)$: $(x+9)+(x+7)=(x+6)+(x+10)$; and $(x+5)+(x+7)=(x+2)+(x+10)$: and (ii) $(5+63)$ and $(8+60)$ are both equal to 68.

So, by this test, at, sight, we know the equation comes under the range of this Sūtra . $4x+22=0$. $x=-5\frac{1}{2}$

Similar is the case with regard to the equation :—

(2) $$\frac{(x+2)\ (x+4)}{(x+1)\ (x+3)}=\frac{(x-1)\ (x+7)}{(x-2)\ (x+6)}$$

$\therefore$ $(x-2)(x+2)(x+4)(x+6)=(x-1)(x+1)(x+3)(x+7)$; and

(i) By cross-addition, the total on both sides is $4x+10$

(ii) The sum of *each* pair of Binomials on the one side is equal to the sum of some pair thereof on the other ; and

(iii) $ab+cd=ef+gh$ i.e. $-4+24=-1+21$ $(=20)$

$\therefore$ The Siitra applies ; and $4x+10=0$ $\quad x=-2\frac{1}{2}$

Such however is not the case with the equation :—

(3) $(x-1)(x-6)(x+6)(x+5)=(x-4)(x-2)(x+3)(x+7)$

Here, we observe :—

(i) The total on both sides is $4x+4$;

but (ii) the totals of pairs of Binomials (on the two sides) do *not* tally ;

and (iii) $ab+cd\neq ef+gh$

This equation is therefore a quadratic (and not within the scope of the present Siitra).

The Algebraical Explanation (for this type of equations) 18 :—

(~ + a(x+b) (~ + G)(x+d)=(x+e) (x+f) (x+g) (x+h)

The data are :—

(i) $a+b+c+d=e+f+g+h$;

(ii) The sum of any pair of binomials on the one side must be the same as the sum of some pair of binomials on the other. Suppose that $a+b=e+f$; and $c+d=g+h$; and

(iii) $ab+cd=ef+gh$

$\therefore$ $x^4+x^3(a+b+c+d)+x^2(ab+ac+ad+bc+bd+cd)$
$+x(abc+abd+acd+bcd)+abcd$
$=x^4+x^3(e+f+g+h)+x^2(ef+eg+eh+fg+fh+gh)$
$+x(efg+efh+egh+fgh)+efgh$

∴ The x^4 and x^3 cancel out; and, owing to the data in the case, the x^2 coefficients are the same on both sides; and thercforc they too cancel out,. And there is no quadratic equation (left for us to solve herein)

Proof: The x^2 coefficients are :—

L.H.S. $ab+ac+ad+bc+bd+cd$

R.H.S. $ef+eg+eh+fg+fh+gh$

i.e. $(ab+cd)+a(c+d)\ b(c+d)=ab+cd+(a+b)(c+d)$

and $(ef+gh)+e(g+h)+f(g+h)=ef+gh+(e+f)(g+h)$

But $(ab+cd)=(ef+gh)$; and $a+b=e+f$; and $c+d=g+h$

∴ the L.H.S.=the R.H.S.; and x^2 vanishes!

FURTHER EXTENSION OF THE SŪTRA

In'the beginning of this very chapter, it was noted that if a function (containing the unknown x, y etc.,) occurs as a common factor in all the terms on both sides (or on the L.H.S. (with zero on the R.H S.) that function can be removed therefrom and equated to Zero. We now proceed to deal with certain types of cases which do not seem to be of this kind but are really so. All that we have to do is to re-arrange the terms in such a manner as to unmask the masked terms, so to say and make the position transparently clear on the surface. For example—

(1) $\frac{x+a}{b+c}+\frac{x+b}{c+a}+\frac{x+c}{a+b}=-3$

Taking -3 over from the R.H.S. to the L.H.S. distributing it amongst the 3 terms there, we have:

$$\frac{x+a}{b+c}+1+\frac{x+b}{c+a}+1+\frac{x+c}{a+b}+1=0$$

i.e. $\frac{x+a+b+c}{b+c}+\frac{x+b+c+a}{c+a}+\frac{x+c+a+b}{a+b}=0$

∴ By virtue of the Samuccaya rule,

$x+a+b+c=0$ ∴ $x=-(a+b+c)$

This whole working can be done, at sight i.e. *mentally.*

(2) $\frac{x+a}{b+c}+\frac{x+b}{c+a}+\frac{x+c}{a+b}=\frac{x+2a}{b+c-a}+\frac{x+2b}{c+a-b}+\frac{x+2c}{a+b-c}$

Add unity to each of the **6** terms ; and observe :—

$$\therefore \frac{x+a+b+c}{D_1}+\frac{x+a+b+c}{D_2}+\frac{x+a+b+c}{D_3}$$
$$=\frac{x+a+b+c}{D_4}+\frac{x+a+b+c}{D_5}+\frac{x+a+b+c}{D_6}$$

$\therefore x+a+b+c=0 \quad \therefore x=-(a+b+c)$

(3) $\frac{x-a}{b+c}+\frac{x-b}{c+a}+\frac{x-c}{a+b}=\frac{x+a}{2a+b+c}+\frac{x+b}{2b+c+a}+\frac{x+c}{2c+a+b}$

Subtract unity from each of the 6 terms ; arid we have :

$x-a-b-c=0 \quad \therefore x=(a+b+c)$

(4) $\frac{x+a^2}{(a+b)(a+c)}+\frac{x+b^2}{(b+c)(b+a)}+\frac{x+c^2}{(c+a)(c+b)}$

$=\frac{x-be}{a(b+c)}=\frac{x-ca}{b(c+a)}+\frac{x-ab}{c(a+b)}$

Subtracting 1 from each of the 6 terms, we have :

$x-ab-ac-bc=0 \quad \therefore x=(ab+bc+ca)$

(5) $\frac{x-bc}{b+c}+\frac{x-ca}{c+a}+\frac{x-ab}{a+b}$

$=\frac{x^2+2a^2-bc}{2a+b+c}+\frac{x^2+2b^2-ca}{2b+c+a}+\frac{x^2+2c^2-ab}{2c+a+b}$

Subtracting a from the first terms, b from the second terms and *c* from the third terms (on both sides), we have :

$x-ab-bc-ca=0 \quad \therefore x=ab+bc+ca$

(6) $\frac{x+a^2+2c^2}{b+c}+\frac{x+b^2+2a^2}{c+a}+\frac{x+c^2+2b^2}{a+b}=0$

As $(b-c)+(c-u)+(a-b)=0$, we add $b-c$, $c-a$ and $a-b$ to the first, second and third terms respectively ; and we have :

$x+a^2+b^2+c^2=0 \quad \therefore x=-(a^2+b^2+c^2)$

(7) $\frac{ax+a(a^2+2bc)}{b-c}+\frac{bx+b(b^2+2ca)}{c-a}+\frac{cx+c(c^2+2ab)}{a-b}=0$

As $a(b-c)+b(c-a)+c(a-b)=0$

$\therefore$ We add $a(b-c)$ to the first term, $b(c-a)$ *to* the second and $c(a-b)$ to the last ; and we have :

$t_1=\frac{ax+a(a^2+2bc)}{b-c}+a(b-c)$

$$=\frac{ax+a(a^2+2bc)+a(b-c)^2}{b-c}$$

$$=\frac{ax+a(a^2+b^2+c^2)}{b-c}=\frac{a}{b-c}\{x+(a^2+b^2+c^2)\}$$

Similarly, $t_2=\frac{b}{c-a}\{x+(a^2+b^2+c^2)\}$

and $\quad t_3=\frac{c}{a-b}\{x+(a^2+b^2+c^2)\}=0$

$\therefore\ x+a^2+b^2+c^2=0 \quad \therefore\ x=-(a^2+b^2+c^2)$

(8) $\frac{x+a^3+2b^3}{b-c}+\frac{x+b^3 2c^3}{c-a}+\frac{x+c^3+2a^3}{a-b}$

$=2a^2+2b^2+2c^2+ab+ac+bc$

Splitting the R.H.S. into $(b^a+bc+c^a)+(c^2+ca+a^2)+(a^2+ab+b^a)$, transposing the three parts to the left and combining the first with the first, the second with the second and the third with the third (by way of application of the '*Ādymādyena*' formula), we have :

$$t_1=\frac{x+a^3+2b^3}{b-c}-(b^2+bc+c^2)$$

$$=\frac{x+a^3+2b^3-b^3+c^3}{b-c}=\frac{x+a^3+b^3+c^3}{b-c}$$

Similarly, $t_2=\frac{\text{the same N}}{C-a}$

and $\quad t_3=\frac{\text{the same N}}{a-b}$

$\therefore\ x=-(a^3+b^3+c^3)$

CHAPTER XIII

MERGER TYPE

of

EASY SIMPLE EQUATIONS (by the *Parāvartya*' method)

Having dealt with various sub-divisions under a few special types of simple equations which the *Śūnyam Sāmyasamuccaye* formula helps us to solve easily, we now go on to and take up another special type of simple equations which the *Parāvartya* Sūtra (dealt with already in connection with Division etc) can tackle for us.

This is of what may be described as the MERGER Type; and this too includes several sub-headings under that heading.

The ***first*** type :

The first variety is one in which a number of terms on the left hand side is equated to a single term on the right hand side, in such manner that $N_1+N_2+N_3$ etc., (the sum of the numerators on the left) and (the single numerator on the right) are the same. For instance,

$$\frac{3}{x+1}+\frac{4}{x-2}=\frac{7}{x+3} \quad \text{Here } N_1+N_2 \text{ i.e. } (3+4)=N(\text{i.e. } 7)$$

So the Sūtra applies.

The procedure is one of merging of the R.H.S. fraction into the left, so that only two terms remain. The process is as follows:

As we mean to merge the R.H.S. into the L.H.S., we subtract the independent term of the to-be-merged binomial from the absolute terms in the binomials on the left and multiply those remainders by the numerators of the terms on the left. And the process is complete.

(i) We first put down the two to-be-retained denominators down thus :—

$$\overline{x+1} \qquad x-2$$

(ii) Then, as 3(from the R.H.S) is to be merged, we subtract that 3 from the 1 in the first term, obtain +2 as the remainder, multiply it by the numerator (i.e. 3), get —6 as the product and put that down as the new numerator for our first term.

(iii) And we do the same thing with the second term, obtain —4 as the product and set it down as our Numerator for the 2nd term of the new (i.e. the derived) equation.

(iv) As the work of merging has been completed, we put zero on the right hand side. So the resultant new equation (after the merger) now reads:

$$\frac{-6}{x+1}-\frac{4}{x+2}=0$$

Then, by simple cross-multiplication, we say

$$\therefore 4x+4=-6x-12 \quad \therefore 10x=-16 \quad \therefore x=-8/5$$

or, by the general formula $(-mb-na)/(m+n)$ explained already (in the chapter on simple equations and first principles), we say at once:

$$x=\frac{12+4}{-6-4}=-8/5$$

The Algebraical Proof hereof is :—

$$\frac{3}{x+1}+\frac{4}{4+2}=\frac{7}{x+3}=\frac{3}{x+3}+\frac{4}{x+3}$$

$$\therefore \frac{3}{x+1}-\frac{3}{x+3}=\frac{4}{x+3}-\frac{4}{x+2}$$

$$\therefore \frac{3(x+3-x-1)}{(x+1)(x+3)}=\frac{4(x+2-x-3)}{(x+3)(x+2)}$$

$$\therefore \frac{6}{x+1}=\frac{-4}{x+2} \quad \therefore 6x+12=4x-4 \quad \therefore 10x=-16$$

$$\therefore x=-8/5$$

The General *Algebraical* Proof hereof is:

$$\frac{p}{x+a}+\frac{q}{x+b}=\frac{p+q}{x+c}$$

$$\therefore \frac{p}{x+a}+\frac{q}{x+b}=\frac{p}{x+c}+\frac{q}{x+c}$$

$$\therefore \frac{p}{x+a}-\frac{p}{x+c}=\frac{q}{x+c}-\frac{q}{x+b}$$

$$\therefore \frac{p(x+c-x-a)}{(x+a)\,(x+c)}=\frac{q(x+b-x-c)}{(x+c)\,(x+b)}$$

$$\therefore \frac{p(c-a)}{x+a}=\frac{q(b-c)}{x+b}$$

$$\therefore x\{p(c-a)+q(c-b)\}=bp(a-c)+aq(b-c)$$

$$\therefore x=\frac{bp(a-c)+aq(b-c)}{p(c-a)+q(c-b)}$$

Well ; the **Algebraical** explanation, may look frightfully long. But the application of the *'Parcivartya'* **Sūtra** (as just herein-above explained and illustrated) is simple enough and easy enough and should be welcomed by the student with delight.

A few more examples of this sort may be noted :

(1) $\frac{3}{x-2}+\frac{5}{x-6}=\frac{8}{x+3}$

Here $3+5=8$ $\therefore$ The Sūtra applies.

$\therefore \frac{(3)(-5)}{x-2}+\frac{(-9)(5)}{x-6}=\frac{-15}{x-2}+\frac{-45}{x-6}=0 \;\therefore x=\frac{-90-90}{-15-45}=3$

(2) $\frac{2}{x+2}+\frac{3}{x+3}=\frac{5}{x+5}$

Here $2+3=5$ $\therefore$ The formula applies.

$\therefore \frac{-6}{x+2}+\frac{-6}{x+3}=0 \;\therefore\; x=-2\frac{1}{2}$

Note :—At this stage. when both the numerators are found to be —6 and can therefore be removed, the formula *"Śūnyaṃ Samuccaye"* may be readily applied : and we may say :

$(x+2)+(x+3)=0 \;\therefore x=-2\frac{1}{2}$

But, if $\frac{2}{x+2}+\frac{3}{x+3}=\frac{5}{x+7}$, as $2+3=$ j.

the merger Sftra applies ; but after the merger. the numerators are different (i.e. -10 and -12) and therefore the *'Śūnyam'* Siitra will not apply.

DISGUISES

Here too, we have often to deal with disguises, by seeing through and penetrating them, in the same way as in the previous chapter (with regard to the *'Śūnyaṃ Samuccaye'* formula).

A few illustrations will make this clear :

(1) $\frac{5}{x-2}+\frac{2}{3-x}=\frac{3}{x-4}$

Here, mere transposition will do the trick. Thus :

$\frac{2}{x-3}+\frac{3}{x-4}=\frac{5}{x-2}$ Now, 2+3 = 5 ∴ The Sūtra applies.

$\therefore \frac{-2}{x-3}+\frac{-6}{x-4}=0 \therefore x=\frac{-26}{-8}=\frac{13}{4}$

(2) $\frac{4}{2x+1}+\frac{9}{3x+2}=\frac{15}{3x+1}$

$4+9\neq15$: Doubt arises ; but the coefficients of x being different in the three denominators, we try the L.C.M. method and get :

$\frac{12}{6x+3}+\frac{18}{6x+4}=\frac{30}{6x+2}$

And here, on noting N_1 (12)+N_2 (18)=N_3 (30), we say : "YES ; the Sūtra applies" and proceed to apply it :

$\therefore 12/(6x+3)+36/(6x+4)=0 \quad \therefore x=-13/24$

But how should we know ***before-hand*** that the Sūtra ***does*** apply ? The TEST is very simple and merely consists in the division of each numerator by the x—coefficient in the denominator (as in the *'Śūnyam'* case). Thus $\frac{4}{2}+\frac{9}{3}=2+3=5$; and $\frac{15}{3}$: is also 5.

Say, "YES" and go ahead, with the merging.

(3) $\frac{4}{2x-1}+\frac{9}{3x-1}=\frac{25}{5x-1}$

Here $(\frac{4}{2}+\frac{9}{3})$ and $\frac{25}{5}$ are the same (i.e. 5) ∴ YES.

$\therefore \frac{60}{30x-15}+\frac{90}{30x-10}=\frac{150}{30x-6}$ NOTE, 60+90=150

$\therefore \frac{2}{30x-15}+\frac{3}{30x-10}=\frac{5}{30x-6}$ NOTE 2+3=5 ∴ YES

Proceed therefore end say:

$$\frac{-18}{30x-15}-\frac{12}{30x-10}=0 \therefore x=\frac{2}{5}$$

(4) $\frac{7}{7x+1}+\frac{6}{4x+1}=\frac{15}{6x+5}$

Here $\frac{7}{7}+\frac{6}{4}=1+1\frac{1}{2}=2\frac{1}{2}$; and $\frac{15}{6}$ is also $2\frac{1}{2}$ $\therefore$ **YES**

Do the merging therefore and say:

$$\therefore \frac{84}{84x+12}+\frac{126}{84x+21}=\frac{210}{84x+70}$$

$$\therefore \frac{2}{84x+12}+\frac{3}{84x+21}=\frac{5}{84x+70} \quad \therefore \text{YES}$$

$$\therefore \frac{-116}{84x+12}+\frac{-147}{84x+21}=0 \quad \therefore x=\frac{-50}{263}$$

(5) $\frac{4}{5x+1}+\frac{7}{10x+1}=\frac{3}{2x+1}$

Here $\frac{4}{5}+\frac{7}{10}=\frac{15}{10}=3/2$ $\therefore$ YES

$$\therefore \frac{8}{10x+2}+\frac{7}{10x+1}=\frac{15}{10x+5} \quad \therefore \text{YES}$$

$$\therefore \frac{-24}{10x+2}+\frac{-28}{10x+1}=0$$

$$\therefore -520x-80=0$$

$$\therefore x=-2/13$$

(6) $\frac{7}{7x+1}+\frac{6}{4x+1}=\frac{15}{6x+1}$

Here $\frac{7}{7}+\frac{6}{4}=2\frac{1}{2}$; and $\frac{15}{6}$ is also $2\frac{1}{2}$ $\therefore$ YES

$$\therefore \frac{84}{84x+12}+\frac{126}{84x+21}=\frac{210}{84x+14} \quad \therefore \text{YES}$$

$$\therefore \frac{2}{84x+12}+\frac{3}{84x+21}=\frac{5}{84x+14} \quad \therefore \text{YE}$$

$$\therefore \frac{4}{84x+12}+\frac{21}{94x+21}=0$$

$$\therefore 1428x+168=0$$

$$\therefore x=\frac{-168}{1428}=\frac{-2}{17}$$

EXTENSION OF MERGER METHOD

(Multiple Merger)

We now take up and deal with equations wherein $N_1+N_2+N_3$(of the L.H.S.)$=N$ of the R.H.S. and wherein the same '*Parāvartya*' (Merger) formula can be applied in exactly the same way as before. Thus:

(1) $\frac{1}{x+2}+\frac{3}{x+3}+\frac{5}{x+5}=\frac{9}{x+4}$

TEST: $1+3+5=9$ ∴ YES.

$\therefore \frac{-2}{x+2}+\frac{-3}{x+3}+\frac{5}{x+5}=0$ ∴ YES, again.

i.e. $\frac{2}{x+2}+\frac{3}{x+3}=\frac{5}{x+5}$

$\therefore \frac{-6}{x+2}+\frac{-6}{x+3}=0$

∴ (i) By the Basic Formula $\left(x=\frac{-mb-na}{m+n}\right)$

$$x=\frac{18+12}{-6-6}=\frac{30}{-12}=-2\tfrac{1}{2}$$

or (ii) By '*Śūnyaṃ Samuccaye*' formula:

$$(x+2)+(x+3)=0 \therefore x=-2\tfrac{1}{2}$$

Note:—These two steps (of successive merging) can be combined into one by multiplying N_1 first by (2—4) and then by (2—5) i.e. by 6 and similarly N_2 first by (3—4) and then by (3—5) i.e. by 2 and proceeding as before

$\therefore \frac{6}{x+2}+\frac{6}{x+3}=0$

∴ By either method (Basic or *Śūnyam*), $x=-2\tfrac{1}{2}$

The Algebraic Proof hereof is this:

$$\frac{m}{x+a}+\frac{n}{x+b}+\frac{p}{x+c}=\frac{m+n+p}{x+d}$$

$$\therefore \frac{m(a-d)}{x+a}+\frac{n(b-d)}{x+b}+\frac{p(c-d)}{x+c}=0$$

$$\therefore \frac{m(a-d)(a-c)}{x+a}+\frac{n(b-d)(b-c)}{x+b}=0$$

which is the exact shape of the formula required for the single-step merger. (vide supra).

Similarly, the merger-formula can be extended to any number of terms as follows:

$$\frac{m}{x+a}+\frac{n}{x+b}+\frac{p}{x+c}+\frac{q}{x+d}+\frac{r}{x+e}+\ldots$$
$$=\frac{m+n+p+q+r+\ldots}{x+w}$$
$$\therefore \frac{m(a-w)(- - -)(a-e)(a-d)(a-c)}{x+a}$$
$$+\frac{n(b-w)(- - -)(b-e)(b-d)(b-c)}{x+b}$$

(which is the general formula for the purpose). Thus, in the above example—

$$x=\frac{(-3)(-2)(-3)+(-3)(-2)(-2)}{(-3)(-2)(+1)+3(-1)(-2)}=\frac{-18-12}{6+6}=\frac{-30}{12}=-2\tfrac{1}{2}$$

A few more illustrations of this type are given below:

(1) $\frac{3}{3x+1}+\frac{4}{4x+1}+\frac{48}{8x+1}=\frac{48}{6x+1}$

Here $\frac{3}{3}+\frac{4}{4}+\frac{48}{8}=8$; and $\frac{48}{6}=8$ $\therefore$ YES

$\therefore \frac{24}{24x+8}+\frac{24}{24x+6}+\frac{144}{24x+3}=\frac{192}{24x+4}$ $\therefore$ YES

$\therefore \frac{1}{24x+8}+\frac{1}{24x+6}+\frac{6}{24x+3}=\frac{8}{24x+4}$ $\therefore$ YES

$\therefore \frac{4}{24x+8}+\frac{2}{24x+6}=\frac{6}{24x+3}$ $\therefore$ YES

$\therefore \frac{20}{24x+8}+\frac{6}{24x+6}=0$

$\therefore 624x+168=0 \quad \therefore x=\frac{-7}{26}$

(2) $\frac{2}{2x+1}+\frac{18}{3x+1}+\frac{75}{5x+1}=\frac{88}{4x+1}$

Here $\frac{2}{2}+\frac{18}{3}+\frac{75}{5}=22$; and $\frac{88}{4}$ is also 22

$\therefore \frac{60}{60x+30}+\frac{360}{60x+20}+\frac{900}{60x+12}=\frac{1320}{60x+15}$ $\therefore$ YES

$\therefore \frac{1}{60x+30}+\frac{6}{60x+20}+\frac{15}{60x+12}=\frac{22}{60x+15}$ $\therefore$ YES

$$\therefore \frac{15}{60x+30}+\frac{30}{60x+20}=\frac{45}{60x+12} \quad \therefore \text{ YES}$$

$$\therefore \frac{1}{60x+30}+\frac{2}{60x+20}=\frac{3}{60x+12} \quad \therefore \text{ YES}$$

$$\therefore \frac{18}{60x+30}+\frac{16}{60x+20}=0$$

$$\therefore 2040x+840=0 \quad \therefore \ x=\frac{-7}{17}$$

Note:—Any change of SEQUENCE (of the terms on the L.H.S.) will cause no change in the working or the result.

(3) $\frac{4}{2x-1}+\frac{27}{3x-1}+\frac{125}{5x-1}=\frac{144}{4x-1}$

Here $\frac{4}{2}+\frac{27}{3}+\frac{125}{5}=2+9+25=36$;

and $\frac{144}{4}$ is also 36 $\therefore$ YES

$$\therefore \frac{120}{60x-30}+\frac{540}{60x-20}+\frac{1500}{60x-12}=\frac{2160}{60x-15} \quad \therefore \text{ YES}$$

$$\therefore \frac{2}{60x-30}+\frac{9}{60x-20}+\frac{25}{60x-12}=\frac{36}{60x-15} \quad \therefore \text{ YES}$$

$$\therefore \text{(By merger)} \frac{-30}{60x-30}+\frac{-45}{60x-20}=\frac{-75}{60x-12} \quad \ldots \text{ YES}$$

$$\therefore \frac{2}{60x-30}+\frac{3}{60x-20}=\frac{5}{60x-12} \quad \therefore \text{ YES}$$

$$\therefore \frac{-36}{60x-30}+\frac{-24}{60x-20}=0$$

$$\therefore \frac{1}{20x-10}+\frac{1}{30x-10}=0$$

$\therefore$ (By Basic rule or by cross-multiplication or by ('*Śūnyam* Formula), $50x-20=0 \ \therefore \ x=\frac{2}{5}$

OR (by Multiple simultaneous merger)

$$60x=\frac{(-40)(-18)(-15)+(-270)(-8)(-5)}{(-2)(-18)(-5)+(9)(-8)(-5)}=24$$

$\therefore x=\frac{2}{5}$

Note:—Again any change of SEQUENCE (of the terms on the L.H.S.) will cause no change in the working or the result.

CHAPTER XIV

COMPLEX MERGERS

There is still another type—a special and complex type of equations which are usually dubbed 'harder' but which can be readily tackled with the aid of the *Parcivartya Sūtra*. For instance :

$$\frac{10}{2x+1}+\frac{3}{3x-2}=\frac{2}{2x-3}+\frac{15}{3x+2}$$

Note the TESTS : (1) $\frac{10}{2}-\frac{3}{3}=\frac{2}{2}+\frac{15}{3}$; and
(2) $10 \times 3=2 \times 15$

i.e. $10 : 15 :: 2 : 3$ (or $10 : 2 :: 15 : 3$)

Transposing. $\frac{10}{2x+1}-\frac{15}{3x+2}=\frac{2}{2x-3}-\frac{3}{3x-2}$,

and taking the L.C.M.

$$\therefore \frac{30}{6x+3}-\frac{30}{6x+4}=\frac{6}{6x-9}-\frac{6}{6x-4}$$

Simple CROSS-MULTIPLICATION leads us to the main TEST :

$$\therefore \frac{30}{(6x+3)(6x+4)}=\frac{30}{(6x-9)(6x-4)}$$

Here comes the third TEST i.e. that the numerator (of the final derived equation) is the same on both sides—

$$\therefore (6x+3)(6x+4)=(6x-9)(Ox-4)$$

$$\therefore 6x=\frac{36-12}{3+4+9+4}=\frac{24}{20}=\frac{6}{5} \therefore x=1/5$$

CLUE—This gives us the necessary clue, namely, that, after putting up the L.C.M. coefficient for x in all the denominators. $(D_1)(D_2)=(D_3)(D_4)$. As the transposition, the L.C.M. etc., can be done ***mentally,*** this clue amounts to a solution of the equation *at **sight.***

In these examples, we should transpose the 4 fractions in such a manner that, after the cross-multiplication etc., are over, all the four denominators (of the final derived equation)

have the same **(L.C.M.)** coefficient for **x** and the numerator is the same on the **L.H.S.** and the R.H.S (of the same equation).

A few more illustrations will be found helpful:

(1) $\frac{2}{6x+1}+\frac{2}{2x-1}=\frac{9}{9x-5}+\frac{1}{3x+1}$

(i) Transposing etc., we have:

$$\frac{6}{18x+3}-\frac{6}{18x+6}=\frac{18}{18x-10}-\frac{18}{18x-9}$$

Here the N on both sides (of the final derived equation) is 18 ∴ The Sūtra applies.

$$\therefore (18x+3)(18x+6)=(18x-10)(18x-9)$$

$$\therefore 18x=\frac{90-18}{3+6+10+9}=\frac{72}{28} \quad \therefore x=\frac{4}{28}=\frac{1}{7}$$

Note:—In some cases (details of which we need not now enter into but which will be dealt with later), the original fractions themselves (after the transposition) fulfil the conditions of the Test. In such cases, we need not bother about the L.C.M. etc., but may straightaway transpose the terms and apply the *'Parcivartya'* formula. In fact, the case just now dealt with is of this type, as will be evident from the following:

(ii) $\frac{2}{6x+1}-\frac{1}{3x+1}=\frac{9}{9x-5}-\frac{2}{2x-1}$

Mere $\frac{2}{6}=\frac{1}{3}$; $\frac{9}{9}=\frac{2}{2}$; and the numerator (on both sides of the final derived equation) is 1.

∴The Sūtra applies and can be applied immediately (without bothering about the L.C.M. etc.).

$\therefore (6x+1)(3x+1)=(9x-5)(2x-1)$

$\therefore 18x^2+9x+1=18x^2-19x+5 \quad \therefore 28x=4 \quad \therefore x=\frac{1}{7}$

(2) $\frac{2}{2x+3}+\frac{3}{3x+2}=\frac{1}{x+1}+\frac{6}{6x+7}$

$$\therefore \frac{2}{2x+3}-\frac{1}{x+1}=\frac{6}{6x+7}-\frac{3}{3x+2}$$

(i) By L.C.M. method, $(6x+9)(6x+6)=(6x+7)(6x+4)$

$\therefore 6x=\frac{-26}{4} \quad \therefore x=\frac{-13}{12}$

(ii) In this case, there is another peculiarity i.e. that the transposition may be done in the other way too and yet the conditions are satisfied. So, we have :

$(6x+9)(6x+7)=(6x+6)(6x+4)$

$\therefore 6x=\frac{-39}{6} \therefore x=\frac{-13}{12}$

(iii) And even, by CROSS-multiplication at the very outset, we get $12x+13=0$ (by *Sāmya Samuccaye*). $\therefore x=\frac{-13}{12}$
In such cases, SEQUENCE (in transposition) does not matter ! (This will be explained later).

(3) $\frac{51}{3x+5}-\frac{68}{4x+11}=\frac{52}{4x-15}-\frac{39}{3x-7}$

TESTS : $\frac{51}{3}$ and $\frac{68}{4}$ are both 17 ; and $\frac{52}{4}$ and $\frac{39}{3}$ are both 13.

This equation can be solved in several ways (all of them very simple and easy) :

(i) By the L.C.M. process :

$$\frac{204}{12x+20}-\frac{204}{12x+33}=\frac{156}{12x-45}-\frac{156}{12x-28}$$

$\therefore$ In the derived equation (in its final form),

$N_1=204\times13=12\times13\times17$;

and $N_2=156\times17=12\times13\times17$

$\therefore$ The Siitra applies.

$\therefore (12x+20)(12x+33)=(12x-45)(12x-28)$

$\therefore 12x=\frac{28\times45-20\times33}{20+33+45+28}=\frac{600}{126} \therefore x=\frac{25}{63}$

(ii) or, removing the common factor (12) :

$$\therefore \frac{17}{12x+20}-\frac{17}{12x+33}=\frac{13}{12x-45}-\frac{13}{12x-28}$$

$\therefore$ In the (final) derived Equation,

$N_1=17\times13$; and $N_2=13\times17$ $\therefore$ The Siitra applies.

$\therefore D_1\times D_2=D_3\times D_4 \therefore 12x=\frac{600}{126} \therefore x=\frac{25}{63}$

(iii) or, at the very outset (i.e. without L.C.M. etc.) :

$$\frac{51}{3x+5}-\frac{68}{4x+11}=\frac{52}{4x-15}-\frac{39}{3x-7}$$

$\therefore$ L.H.S. $N=561-340=221$; and

R.H.S. $N=-364+585=221$

$\therefore$ The **Sūtra** applies straightaway.

$\therefore (3x+5)(4x+11)=(4x-15)(3x-7)$

$\therefore 12x^2+53x+55=12x^2-73x+105$

$\therefore 126x=50 \therefore x=\frac{25}{63}$

Note:—In the second method, note that

$N_1=N_2=D_4-D_3$ and $N_3=N_4=D_2-D_1$ TESTS

The General Formula applicable in such cases is:

$$\frac{m-n}{x+p}+\frac{p-q}{x+n}=\frac{m-n}{x+q}+\frac{p-q}{x+m}$$

$$\therefore (m-n)\left(\frac{1}{x+p}-\frac{1}{x+q}\right)=(p-q)\left(\frac{1}{x+m}-\frac{1}{x+n}\right)$$

$$\therefore \frac{(m-n)(q-p)}{(x+p)(x+q)}=\frac{(p-q)(n-m)}{(x+m)(x+n)}$$

As the numerators are the same,

$\therefore$ The **Sūtra** applies

$\therefore (x+p)(x+q)=(x+m)(x+n)$

$$\therefore x=\frac{mn-pq}{p+q-m-n}$$

(4) $\frac{1}{2x-1}+\frac{8}{4x-1}=\frac{6}{3x-1}+\frac{3}{6x-1}$

(i) $\therefore \frac{6}{12x-6}-\frac{6}{12x-2}=\frac{24}{12x-4}-\frac{24}{12x-3}$

$\therefore$ In the **final** derived equation,

L.H.S. $N=24$; and R.H.S. N is also 24

$\therefore$ The **Sūtra** applies.

$\therefore 12x=\frac{12-12}{D}=0 \therefore x=0$

(ii) '*Vilokana*' (i.e. mere observation) too will suffice in this case.

(5) $\frac{3}{3x+1}+\frac{2}{2x-1}=\frac{3}{3x-2}+\frac{2}{2x+1}$

(i) Here the resultant N is the same (1) (on both sides)

$\therefore$ YES

$\therefore 6x^2+5x+1=6x^2-7x+2 \therefore 12x=1 \therefore x=\frac{1}{12}$

(ii) or, by cross-multiplication at the very outset and *Śūnyaṃ Samuccaye*, $12x-1=0 \therefore x=\frac{1}{12}$

(6) $\frac{5}{5x+2}+\frac{3}{3x+1}=\frac{5}{5x+3}+\frac{15}{15x+2}$

(i) . $\frac{15}{15x+6}-\frac{15}{15x+9}=\frac{16}{15x+2}-\frac{15}{15x+5}$

$\therefore$ The resultant Numerator on both sides is 45

$\therefore$ The Sūtra applies.

$\therefore 15x=\frac{10-54}{8}=\frac{-44}{8} \therefore x=\frac{-11}{30}$

(ii) Or, by cross-multiplication at the very outset and *Śūnyam* etc., formula, we get $30x+11$ and $150x+55$ on the L.H.S. and the R.H.S. respectively; and the uumerical factor (5) being removed, both give us $30x+11=0 \therefore x=\frac{-11}{30}$

(7) $\frac{2x+11}{x+5}+\frac{6x+11}{2x+3}=\frac{4x+4}{2x+1}+\frac{3x+19}{x+6}$

(i) $\because$ (By *Parāvartya* division):

$\frac{1}{x+5}+\frac{2}{2x+3}=\frac{2}{2x+1}+\frac{1}{x+6}$

$\therefore \frac{2}{2x+10}-\frac{2}{2x+12}=\frac{2}{2x+1}-\frac{2}{2x+3}$

$\therefore$ 4 is the N on both sides (of the derived equation)

$\therefore$ The Sūtra applies.

$\therefore (2x+10)(2x+12)=(2x+1)(2x+3)$

$\therefore 2x=\frac{3-120}{18}=\frac{-117}{18} \therefore x=\frac{-13}{4}$

(ii) or by cross-multiplication at the very outset and *Śūnyam Sūtra*, we have:

$4x+13=0 \therefore x=\frac{-13}{4}$

(8) $\frac{2x+11}{x+5}+\frac{15x-47}{3x-10}=\frac{9x-9}{3x-4}+\frac{4x+13}{x+3}$

(i) $\therefore \frac{3}{3x+15}+\frac{3}{3x-10}=\frac{3}{3x-4}+\frac{3}{3x+9}$

$\therefore \frac{3}{3x+15}-\frac{3}{3x+9}=\frac{3}{3x-4}-\frac{3}{3x-10}$

In the resultant equation,

—18 is the numerator on both sides

$\therefore$ The Sūtra applies.

$\therefore (3x+15)(3x+9)=(3x-4)(3x-10)$

$\therefore 3x=\frac{40-135}{38}=\frac{-5}{2} \therefore x=\frac{-5}{6}$

(ii) or by cross-multiplication at the very outset and *Śūnyam* formula,

$\therefore 18x+15=0 \therefore x=\frac{-5}{6}$

(9) $\frac{12x^2+9x+7}{3x+4}+\frac{12x^2+x+3}{4x-1}=\frac{24x^2+14x+3}{12x+1}+\frac{5x^2+6x+2}{x+1}$

$\therefore$ By (*Parāvartya*) devision twice over.

$\frac{3}{3x+4}+\frac{4}{4x-1}=\frac{12}{12x+1}+\frac{1}{x+1}$

$\frac{12}{12x+16}+\frac{12}{12x-3}=\frac{12}{12x+1}+\frac{12}{12x+12}$

$\therefore$ By '*Śūnyam*' Sūtra, we immediately obtain:

$24x+13=0 \therefore x=\frac{-13}{24}$

Note :—The Cross-multiplication and '*Śūnyam*' method is so simple, easy and straight before us here that there is no need to try any other process at all. The student may, however, for the sake of practice try the other methods also and get further *verification* therefrom for the correctness of the answer just hereinabove arrived at.

CHAPTER XV

SIMULTANEOUS SIMPLE EQUATIONS

Here too, we have the GENERAL FORMULA applicable to all cases (under the '*Parāvartya*' Sūtra) and also the special Sūtras applicable only to special types of cases.

THE GENERAL FORMULA

The current system may congratulate and felicitate itself on having a fairly satisfactory method—known as the Cross-multiplication method—for the solving of simultaneous simple equations, which is somewhat akin to the Vedic *'Parcivartya'* method and comes very near thereto.

But even here, the unfortunate drawback still remains that, in spite of all the arrow-directions etc., intended to facilitate its use, the students (and sometimes even the teachers) of Mathematics often get confused as regards the plus and the *minus* signs (+ and −) and how exactly they should be used; and, consequently, we find most of them preferring—in actual daily practice—the substitution method or the elimination method (by which they frame new equations involving only x or only y). And this, of course, does not permit a one-line mental-method answer; and it entails the expenditure of more time and more toil.

The Vedic method (by the *Parāvartya* Rule) enables *us* to give the answer immediately (by mere mental Arithmetic). Thus—

$$\left.\begin{aligned} 2x+3y &= 8 \\ 4x+5y &= 14 \end{aligned}\right\}$$

The rule followed is the "Cyclic" one:

(i) For the value of x, we start with the y-coefficients and the independent terms and cross-multiply forward (i.e. rightward) (i.e. we start from the upper row and multiply

across by the lower one; and conversely; and the connecting link between the two cross-prodcts is always a *minus*). And this gives us our Numerator;

(ii) For finding the Denominator. we go from the upper row across to the lower one (i.e. the x coefficient) but backward (i.e. leftward). Thus,

$$\left.\begin{aligned}2x+3y&=8\\4x+5y&=14\end{aligned}\right\}$$ ∴ for the value of x, the numerator is $3\times14-5\times8=2$; and the Denominator is $8\times4-2\times5=2$

In other words $x=\frac{2}{2}=1$.

And, as for the value of y, we follow the cyclic system (i.e. start with the independent term on the upper row towards the x coefficient on the lower row). So, our Numerator is:

$$8\times4-14\times2=32-28=4$$

And NOTE that the Denominator is invariably the SAME as before (for x) and thus we avoid the confusion caused in the current system by another set of multiplications, a change of sign etc. In other words, $y=\frac{4}{2}=2$

(2) $\left.\begin{aligned}x-y&=7\\5x+2y&=42\end{aligned}\right\}$ $\therefore x=\frac{-42-14}{-5-2}=\frac{-56}{-7}=8;$ and $y=\frac{35-42}{-7}=\frac{-7}{-7}=1$

(3) $\left.\begin{aligned}2x+y&=5\\3x-4y&=2\end{aligned}\right\}$ $\therefore x=\frac{2+20}{3+8}=\frac{22}{11}=2;$ and $y=\frac{15-4}{11}=\frac{11}{11}=1;$

(4) $\left.\begin{aligned}5x-3y&=11\\6x-5y&=9\end{aligned}\right\}$ $\therefore x=\frac{-27+55}{-18+25}=\frac{28}{7}=4;$ and $y=\frac{66-45}{7}=\frac{21}{7}=3$

(5) $\left.\begin{aligned}11x+6y&=28\\7x-4y&=10\end{aligned}\right\}$ $\therefore x=\frac{60+112}{42+44}=\frac{172}{86}=2;$ and $y=\frac{196-110}{86}=\frac{86}{86}=1$

A SPECIAL TYPE

There is a special type of simultaneous simple equations which may involve big numbers and may therefore seem "hard" but which, owing to a certain ratio between the coefficients, can be readily i.e. mentally solved with the aid of the Sūtra आनुरूप्ये शून्यं अन्यत् (*Sūnyaṃ Anyat*) (which cryptically says : If one is in ratio, the other one is Zero).

An example will make the meaning and the application clear :

$$\left.\begin{array}{r} 6x+7y=8 \\ 19x+14y=16 \end{array}\right\}$$

Here we note that the y-coefficients are in the same ratio to each other as the independent terms are to each other. And the Sūtra says that, in such a case, the other one, namely, $x=0$. This gives us two simple equations in y, which give us the same value $\frac{8}{7}$ for y. Thus $x=0$; $y=\frac{8}{7}$

N.B. :—Look for the ratio of the coefficients of *one* of the unknown quantities being the same as that of the independent terms (on the R.H.S.) ; and if the four are in proportion, put the other unknown quantity down as zero ; and equate the first unknown quantity to the absolute term on the right.

The Algebraical Proof is this:

$$\left.\begin{array}{r} ax+by=bm \\ cx+dy=dm \end{array}\right\}$$

$$\therefore \left.\begin{array}{r} adx+bdy=bdm \\ bcx+bdy=bdm \end{array}\right\}$$

$$\therefore x(ad-bc)=0 \quad \therefore \left.\begin{array}{r} x=0 \\ \text{and } y=m \end{array}\right\}$$

A few more illustrations may be taken:

(1) $\left.\begin{array}{r} 12x+8y=7 \\ 16x+16y=14 \end{array}\right\}$ Here, $\because 8:16::7:14$ (mentally)

$$\therefore \left.\begin{array}{r} x=0 \\ \text{and } y=\frac{7}{8} \end{array}\right\}$$

(2) $\left.\begin{array}{l}12x+78y=12\\16x+96y=16\end{array}\right\}$ Here ∵ 12 : 16 :: 12 : 16 (mentally)
∴ x = 1
and y = 0

(3) $\left.\begin{array}{r}499x+172y=212\\9779x+387y=477\end{array}\right\}$

Here 172 = 4 X 43 and 387 = 9 X 43 } ∴ The ratio is the
and 212 = 4 × 53 and 477 = 9 × 53 } same

∴ x = 0 }
and y = $\frac{53}{43}$ }

Note:—The big coefficients (of x=01) need not frighten us!)

N.B.:—This rule is also capable of infinite extension and may be extended to any number of unknown quantities.

Thus:

(1) $\begin{array}{l}ax+by+cz=a\\bx+cy+az=b\\cx+ay+by=c\end{array}$ $\left.\begin{array}{r}\therefore x=1\\y=0\\\text{and } z=0\end{array}\right\}$

(2) $\left.\begin{array}{r}ax+by+cz=cm\\ax+ay+fz=fm\\mx+py+qz=qm\end{array}\right\}$ $\left.\begin{array}{r}\therefore x=0\\y=0\\\text{and } z=m\end{array}\right\}$

(3) $\left.\begin{array}{l}97x+ay+43z=am\\49979x+by+(p+q)z=bm\\49x(a-d)^3+cy+(m-n)^8z=cm\end{array}\right\}$ $\begin{array}{r}\therefore x=0\\y=0\\\text{and } z=m\end{array}$

N.B. :—The coefficients have been deliberately made big and complex but need not frighten us.

A Second Special Type

There is another special type of simultaneous linear equations where the x-coefficients and the y-coefficients are found interchanged. No elaborate multiplications etc., are needed here. The (axiomatic) *Upasūtra* संकलन-व्यवकलनाभ्यां (*'Saṅkalana-Vyavakalanābhyām'*) (which means "By addition and by subtraction) gives us immediately two equations giving the values of (x+y) and (x−y). And a repetition of the same

process gives us the values of x and y ! And the whole work can be done mentally. Thus:

$$\left.\begin{array}{l}45x-23y=113\\23x-45y=\ \ 91\end{array}\right\}$$

$\therefore$ By addition, $68x-68y=68(x-y)=204$ $\therefore$ $x-y=3$
And by subtraction, $22x+22y=22(x+y)=22$ $\therefore$ $x+y=1$

$$\left.\begin{array}{l}\therefore\ x=2\\\text{and }\ y=-1\end{array}\right\}$$

Note:—However big and complex the coefficients may be, there is no multiplication involved but only simple addition and simple subtraction.

The other special types of **simultaneously linear** equations will be discussed at **a** later stage.

Chapter XVI

MISCELLANEOUS (SIMPLE) EQUATIONS.

There are other types of miscellaneous linear equations which can be treated by the Vedic Sūtras. A few of them are shown below.

FIRST TYPE

Fractions of a particular cyclical kind are involved here. And, by the ***Parāvartya*** **Sūtra,** we write down the Numerator of the sum-total of all the fractions in question and equate it to zero. **Thus:**

(1) $$\frac{1}{(x-1)(x-2)}+\frac{2}{(x-2)(x-3)}+\frac{3}{(x-3)(x-1)}=0$$

Here, each numerator is to be multiplied by the factor absent from its denominator. This is usually and actually done everywhere but not as a rule of mental practice. This, however, should be regularly practised; and the resultant numerator equated to zero.

In the present instance,

$$(x-3)+(2x-2)+(3x-6)=6x-11=0 \;\therefore\; x=\frac{11}{6}$$

The ***Algebraical*** proof is well-known and is as follows:

$$\frac{p}{(x+a)(x+b)}+\frac{q}{(x+b)(x+c)}+\frac{r}{(x+c)(x+a)}$$

$$=\frac{p(x+c)+q(x+a)+r(x+b)}{(x+a)(x+b)(x+c)}$$

$$=\frac{x(p+q+r)+(pc+qa+rb)}{(x+a)(x+b)(x+c)}=0$$

$$\therefore x=\frac{-(pc+qa+rb)}{p+q+r}$$

In other words,

$$x=\frac{\text{Each N multiplied by the absent number reversed}}{N_1+N_2+N_3}$$

As this is simple and easy to remember and to apply, the work can be done *mentally*. And we can say, $x=\frac{11}{}$

A few more examples are noted :

(1) $\frac{1}{(x-1)(x-3)}+\frac{3}{(x-3)(x-5)}+\frac{5}{(x-5)(x-1)}=0$

$\therefore\ x=\frac{5+3+15}{1+3+5}=\frac{23}{9}$

(2) $\frac{2}{(x-1)(x+2)}+\frac{3}{(x+2)(x-4)}+\frac{4}{(x-4)(x-1)}=0$

$\therefore\ x=\frac{8+3-8}{2+3+4}=\frac{3}{9}=\frac{1}{3}$

(3) $\frac{1}{(x-3)(x-4)}+\frac{3}{(x-4)(x-9)}+\frac{5}{(x-9)(x-3)}=0$

$\therefore\ x=\frac{9+9+20}{1+3+5}=\frac{38}{9}$

A few disguised samples may also be taken :

(1) $\frac{1}{x^2+3x+2}+\frac{5}{x^2+5x+6}+\frac{3}{x^2+4x+3}=0$

$\therefore$ (mentally)

$\frac{1}{(x+1)(x+2)}+\frac{5}{(x+2)(x+3)}+\frac{3}{(x+3)(x+1)}=0$

$\therefore\ x=\frac{-3-5-6}{1+5+3}=\frac{-14}{9}$

(2) $\frac{1}{6x^2+5x+1}+\frac{1}{12x^2+7x+1}+\frac{1}{8x^2+6x+1}=0$

$\therefore$ (mentally)

$\frac{}{(2x+1)(3x+1)}+\frac{1}{(3x+1)(4x+1)}+\frac{1}{(4x+1)(2x+1)}=0$

$\therefore\ 9x+3=0\ \therefore\ x=-\frac{1}{3}$

(3) $\frac{3}{(x+3)^2-2^2}+\frac{2}{(x+4)^2-1^2}+\frac{1}{(x+2)^2-1^2}=0$

$\therefore$ (mentally) $\frac{3}{(x+1)(x+5)}+\frac{2}{(x+5)(x+3)}+\frac{1}{(x+3)(x+1)}=0$

$\therefore\ x=\frac{-9-2-5}{3+2+1}=\frac{-16}{6}=\frac{-8}{3}$

(4) $\frac{x+4}{(x+1)(x+3)}+\frac{x+8}{(x+3)(x+5)}+\frac{x+6}{(x+5)(x+1)}=\frac{3}{x}$

$\therefore$ (mentally)

$\frac{x^2+4x}{(x+1)(x+3)}-1+\frac{x^2+8x}{(x+3)(x+5)}-1+\frac{x^2+6x}{(x+5)(x+1)}-1=0$

$\therefore$ (mentally) $\frac{-3}{(x+1)(x+3)}+\frac{-15}{(x+3)(x+5)}+\frac{-5}{(x+5)(x+1)}=0$

$\therefore\ x=\frac{15+15+15}{-3-15-5}=\frac{-45}{23}$

(5) $\frac{x-3}{(x-1)(x-2)}+\frac{x-5}{(x-2)(x-3)}+\frac{x-4}{(x-3)(x-1)}=\frac{3}{x}$

$\therefore$ (mentally) $\frac{-2}{(x-1)(x-2)}+\frac{-6}{(x-2)(x-3)}+\frac{-3}{(x-3)(x-1)}=0$

$\therefore\ x=\frac{-6-6-6}{-2-6-3}=\frac{18}{11}$

(6) $\frac{x-4}{(x-1)(x-3)}+\frac{x-9}{(x-3)(x-6)}+\frac{x-7}{(x-6)(x-1)}=\frac{3}{x}$

$\therefore$ (mentally) $\frac{-3}{(x-1)(x-3)}+\frac{-18}{(x-3)(x-6)}+\frac{-6}{(x-6)(x-1)}=0$

$\therefore\ x=\frac{-18-18-18}{-3-18-6}=\frac{54}{27}=2$

(7) $\frac{x-6}{(x-2)(x-3)}+\frac{x-8}{(x-3)(x-4)}+\frac{x-7}{(x-4)(x-2)}=\frac{3}{x+1}$

$\therefore$ (mentally) $\frac{-12}{(x-2)(x-3)}-\frac{-20}{(x-3)(x-4)}-\frac{-15}{(x-4)(x-2)}=0$

$\therefore\ x=\frac{-48-40-45}{-12-20-15}=\frac{133}{47}$

(8) $\frac{35x+23}{(5x-1)(7x-1)}+\frac{63x+47}{(7x-1)(9x-1)}+\frac{45x+31}{(9x-1)(5x-1)}=\frac{3}{x-1}$

$\therefore \left\{\frac{35x^2-12x-23}{(5x-1)(7x-1)}-1\right\}+\left\{\frac{63x^2-16x-47}{(7x-1)(9x-1)}-1\right\}+\left\{\frac{45x^2-14x-31}{(9x-1)(5x-1)}-1\right\}=0$

$\therefore \frac{3}{(5x-1)(7x-1)}+\frac{6}{(7x-1)(9x-1)}+\frac{4}{(9x-1)(5x-1)}=0$

$\therefore 85x-13=0 \quad \therefore\ x=\frac{13}{85}$

SECOND TYPE

A second type of such special simple equations is one where we have $\frac{1}{AB}+\frac{1}{AC}=\frac{1}{AD}+\frac{1}{BC}$ and the factors (A, B, C and D) of the denominators are in Arithmetical Progression. The Sūtra

सोपान्त्यद्वयमन्त्यं (*Sopāntyadvayamantyam*) which means "the ultimate and twice the penultimate" gives us the answer immediately, for instance:

$$\frac{1}{(x+2)(x+3)}+\frac{1}{(x+2)(x+4)}=\frac{1}{(x+2)(x+5)}+\frac{1}{(x+3)(x+4)}$$

Here, according to the **Sūtra, L+2p** (the **last+twice** the **penultimate**) $=(x+5)+2(x+4)=3x+13=0 \quad \therefore \; x=-4\frac{1}{3}$

The proof of this is as follows:

$$\frac{1}{(x+2)(x+3)}+\frac{1}{(x+2)(x+4)}=\frac{1}{(x+2)(x+5)}+\frac{1}{(x+3)(x+4)}$$

$$\therefore \frac{1}{(x+2)(x+3)}-\frac{1}{(x+2)(x+5)}=\frac{1}{(x+3)(x+4)}-\frac{1}{(x+2)(x+4)}$$

$$\therefore \frac{1}{x+2}\left\{\frac{2}{(x+3)\,(x+5)}\right\}=\frac{1}{x+4}\left\{\frac{-1}{(x+2)\,(x+3)}\right\}$$

Removing the factors (x+2) and (x+3)

$$\frac{2}{x+5}=\frac{-1}{x+4} \text{ i.e. } \frac{2}{L}=\frac{-1}{P} \therefore L+2P=0$$

The General *Algebraical* Proof is as follows:

$\frac{1}{AB}+\frac{1}{AC}=\frac{1}{AD}+\frac{1}{BC}$ (where A,B, C and D are in A.P.)

Let d be the common difference.

$$\therefore \frac{1}{A(A+d)}+\frac{1}{A(A+2d)}=\frac{1}{A(A+3d)}+\frac{1}{(A+d)(A+2d)}$$

$$\therefore \frac{1}{A(A+d)}-\frac{1}{A(A+3d)}=\frac{1}{(A+d)(A+2d)}-\frac{1}{A(A+2d)}$$

$$\therefore \frac{1}{A}\left\{\frac{2d}{(A+d)(A+3d)}\right\}=\frac{1}{A+2d}\left\{\frac{-d}{A(A+d)}\right\}$$

Cancelling the factors A(A+d) of the denominators and d of the Numerators:

$$\therefore \frac{2}{A+3d}=\frac{-1}{A+2d}$$

In other words $\frac{2}{L}=\frac{-1}{P} \therefore L+2P=0$

Another Algebraical proof. :

$$\frac{1}{AB}+\frac{1}{AC}=\frac{1}{AD}+\frac{1}{BC}$$

$$\therefore \frac{1}{AB}-\frac{1}{AD}=\frac{1}{BC}-\frac{1}{AC}$$

$$\therefore \frac{1}{A}\left\{\frac{D-B}{BD}\right\}=\frac{1}{C}\left\{\frac{A-B}{AB}\right\}$$

But $\because$ A, B, C and D are in AP

$\therefore D-B=-2\,(A-B)$

$\therefore \frac{2}{D}=\frac{-1}{C} \quad \therefore 2C+D=0$; i.e. $2P+L=0$

A few more samples may be tried :

(1) $$\frac{1}{x^2+7x+12}+\frac{1}{x^2+8x+15}-\frac{1}{x^2+9x+18}+\frac{1}{x^a+9x+20}$$

$\therefore$ (mentally)

$$\frac{1}{(x+3)(x+4)}+\frac{1}{(x+3)(x+5)}=\frac{1}{(x+3)(x+6)}+\frac{1}{(x+4)(x+5)}$$

$\therefore 2P+L=(2x+10)+(x+6)=0 \quad \therefore x=-5\frac{1}{3}$

(2) $$\frac{1}{(2x+1)(3x+2)}+\frac{1}{(2x+1)(4x+3)}=\frac{1}{(2x+1)(5x+4)}+\frac{1}{(3x+2)(4x+3)}$$

$\therefore 2P+L=(8x+6)=(5x+4)=13x+10=0 \quad \therefore x=\frac{-10}{13}$

THIRD TYPE

A third type of equations are those where Numerator and Denominator on the L.H.S. (barring the independent terms) stand in the same ratio to each other as the entire Numerator and the entire Denominator of the R.H.S. stand to each other and these can be readily solved with the aid of the **Upasūtra** (subformula or corollary) अन्त्ययोरेव (Antyayoreva) which means, "only the last terms" i.e. the absolute terms. Thus :

$$\frac{x^2+x+1}{x^2+3x+3}=\frac{x+1}{x+3}$$

Here, $(x^2+x)=x\,(x+1)$ and $(x^S+3x)=x(x+3)$

$\therefore$ The Rule applies; and we say :

$$\frac{x+1}{x+3}=\frac{1}{3} \quad \therefore x=0$$

The Algebraical proof is as follows :

$$\frac{AC+D}{BC+E}=\frac{A}{B}=\frac{AC}{BC}=\frac{D}{E} \text{ (by Dividend)}$$

Another Algebraic Proof is this

$$\frac{AC+D}{BC+E}=\frac{A}{B} \quad \therefore ABC+AE=ABC+BD$$

$$\therefore AE=BD \quad \therefore \frac{A}{B}=\frac{D}{E}$$

A few more examples may be taken :

(1) $\frac{3x^2+5x+8}{5x^2+6x+12}=\frac{3x+5}{5x+6}=\frac{8}{12} \quad \therefore 4x=12 \quad \therefore x=3$

(2) $\frac{2-2x-3x^2}{2-5x-6x^2}=\frac{3x+2}{6x+5}=\frac{2}{2} \quad \therefore x=-1$

(3) $\frac{81x^2+108x+2}{54x^2+27x+5}=\frac{3x+4}{2x+1}=\frac{2}{5} \quad \therefore x=\frac{-18}{11}$

(4) $\frac{58x^2+87x+7}{87x^2+145x+11}=\frac{2x+3}{3x+5}=\frac{7}{11} \quad \therefore x=2$

(5) $\frac{158x^2+237x+4}{395x^2+474x+4}=\frac{2x+3}{5x+6}=\frac{4}{4} \quad \therefore x=-1$

(6) $\frac{1-px}{1-qx}=\frac{2+pqx-p^2qx^2}{2+pqx-pq^2x^2}=\frac{2}{2} \quad \therefore x=0$

(7) $\frac{(2x+3)^2}{(2x+5)}=\frac{x+3}{x+5} \quad \therefore \frac{4x^2+12x+9}{4x^2+20x+25}=\frac{x+3}{x+5}=\frac{9}{25} \quad \therefore x=\frac{-15}{8}$

(8) $\frac{(x+1)(x+6)}{(x+3)(x+5)}=\frac{x+7}{x+8}$

$\therefore \frac{x^2+7x+6}{x^2+8x+15}=\frac{x+7}{x+8}=\frac{6}{15} \quad \therefore x=-6\frac{1}{3}$

Note :—By cross-multiplication,

$(x+1)(x+6)(x+8)=(x+3)(x+5)(x+7)$

Here, the total of the Binomials is $3x+15$ on each side. But the *Śūnyam Samuccaye* Sūtra does *not* apply beacuse the number of factors (in the original shape) is 2 on the L.H.S. and only one on the R.H.S. '*Antyayoreva* is the Sūtra to be applied.

(9) $(x+1)(x+2)(x+9)=(x+3)(x+4)(x+5)$

The total (on each side) is the same (i.e. $3x+12$). But the '*Śūnyam Samuccaye*' Sūtra does *not* apply. The '*Antyayoreva*' formula is the one to be applied.

$$\frac{(x+1)(x+2)}{(x+4)(x+5)}=\frac{x+3}{x+9}=\frac{2}{20} \quad \therefore x=\frac{-7}{3}$$

(10) $(x+2)(x+3)(x+11)=(x+4)(x+5)(x+7)$.

The case is exactly like the one above.

$$\therefore \frac{(x+2)(x+3)}{(x+4)(x+7)}=\frac{x+5}{x+11}=\frac{6}{28} \quad \therefore x=\frac{-74}{22}=\frac{-37}{11}$$

FOURTH TYPE

Another type of special Fraction-Additions (in connection with Simple equations) is often met with, wherein the factors of the Denominators are in Arithmetical Progression or related to one another in a special manner as in SUMMATION OF SERIES. These we can readily solve with the aid of the same "*Antyayoreva*" Sūtra [but in a different context, and in a different sense). We therefore deal with this special type here.

(1) The first sub-section of this type is one in which the factors are in AP. Thus:

$$\frac{1}{(x+1)(x+2)}+\frac{1}{(x+2)(x+3)}+\frac{1}{(x+3)(x+4)}$$

The Sūtra tells us that the sum of this aeries is a fraction whose numerator is the sum of the numerators in the series and whose denominator is the product of the two ends i.e. the first and the last Binomials!

So, in this case, $S_3=\frac{3}{(x+1)(x+4)}$ and so on.

The Algebraical proof of this is as follows:

$$t_1+t_2=\frac{1}{(x+1)(x+2)}+\frac{1}{(x+2)(x+3)}=\frac{x+3+x+1}{(x+1)(x+2)(x+3)}$$

$$=\frac{2(x+2)}{(x+1)(x+2)(x+3)}=\frac{2}{(x+1)(x+3)}$$

wherein the Numerator is the sum of the original Numerators and the Denominator is the product of the first and the last Binomial factors.

Adding t_3 to the above, we have $\frac{2}{(x+1)(x+3)}+\frac{1}{(x+3)(x+4)}$

$$=\frac{2x+8+x+1}{(x+1)(x+3)(x+4)}=\frac{3(x+3)}{(x+1)(x+3)(x+4)}=\frac{3}{(x+1)(x+4)}$$

Continuing this process to any number of terms, we find the Numerator continuously increases by one and the Denominator invariably drops the middle binomial and retains only the first and the last, thus proving the correctness of the Rule in question.

$$t_1=\frac{1}{(x+1)(x+2)}=\frac{1}{x+1}-\frac{1}{x+2}$$

$$t_2=\frac{1}{(x+2)(x+3)}=\frac{1}{x+2}-\frac{1}{x+3}$$

and so on to any number of terms

Note:—The second term of each step on the R.H.S. and the first term on the next step (of the L.H.S) cancel each other and that, consequently, whatever may be the number of terms which we take, all the terms (on the R.H.S.) except the very first and the last cancel out and the Numerator (being the difference between the first and the last binomial i.e. the only binomials surviving) is the sum of the original Numerators (on the L.H.S.) .And this proves the proposition in question.

A few more illustrations are taken.

(1) $\frac{1}{(x+3)(x+4)}+\frac{1}{(x+4)(x+5)}+\dots$

$$\therefore S_4=\frac{1}{(x+3)(x+7)}$$

(2) $\frac{1}{x^2-3x+2}+\frac{1}{x^2-5x+6}+\dots$

$$=\frac{1}{(x-1)(x-2)}+\frac{1}{(x-2)(x-3)}+\cdot$$

$$\therefore S_5=\frac{1}{(x-1)(x-6)}$$

(3) $\frac{1}{x^2+5x+4}+\frac{1}{x^2+11x+28}+\dots$

$$=\frac{1}{(x+1)(x+4)}+\frac{1}{(x+4)(x+7)}+\dots$$

$$\therefore S_4=\frac{4}{x^2+14x+13}$$

(4) $\frac{1}{(x+a)(x+2a)}+\frac{1}{(x+2a)(x+3a)}+\dots$

$$\therefore S_4=\frac{4}{(x+a)(x+5a)}$$

(5) $\frac{1}{(x+1)(3x+1)}+\frac{1}{(3x+1)(5x+1)}+\frac{1}{(5x+1)(7x+1)}+\dots$

Here, there is a slight difference in the *structure* of the Denominator i.e. that the A.P. is not in respect of the independent term in the binomials (as in the previous examples) but in the x-coefficient itself But this makes no difference as regards the applicability of the Siitra.

$$\therefore S_3=\frac{3}{(x+1)(7x+1)}$$

The First Algebraical Proof of this is exactly as before:

$$t_1+t_2=\frac{2}{(x+1)(5x+1)};\ t\ +t_2+t_3=\frac{3}{(x+1)(7x+1)}\text{ and so on}$$

The addition of each new term automatically establishes the proposition.

The second Algebraical proof is slightly different but follows the same lines and leads to the same result:

$$t_1=\frac{1}{(x+1)(3x+1)}=\frac{1}{2x}\left(\frac{1}{x+1}-\frac{1}{3x+1}\right)$$

$$t_2=\frac{1}{(3x+1)(5x+1)}=\frac{1}{2x}\left(\frac{1}{3x+1}-\frac{1}{5x+1}\right)\text{ and so on}$$

Note:—The cancellations take place exactly as before, with the consequence that the sum-total of the fractions=

$\frac{1}{2x}\ \frac{(2lcx)}{D_1\times D_l}=\frac{l}{D_1\times D_c}$ (Where l stands for S_N)

(which proves the proposition)

(6) $\frac{1}{(x+a)(2x+3a)}+\frac{1}{(2x+3a)(3x+5a)}+\dots$

Here, the progression is with regard to both the items in the binomials (i.e. the x-coefficients and the absolute terms). But this too makes no difference to the applicability of the formula under discussion.

$$\therefore S_3=\frac{3}{(x+a)(4x+7a)}$$

(7) $$\frac{1}{(3x+a)(5x+a)}+\frac{1}{(5x+a)(7x+a)}+\dots$$

$$\therefore S_3=\frac{3}{(3x+a)(9x+a)}$$

(8) $$\frac{1}{(x^2+x+1)(x^2+2x+2)}+\frac{1}{(x^2+2x+2)(x^2+3x+3)}+\dots$$

Seemingly, there is a still greater difference in the *structure* of the Denominators. But even this makes no difference to the applicability of the aphorism. *So* we say:

$$S_4=\frac{4}{(x^2+x+1)(x^2+5x+5)}$$

Both the Algebraical explanations apply to this case also. And we may extend the rule indefinitely to as many terms and to as many varieties as we may find necessary.

We may conclude this sub-section with a few examples of its application to Arithmetical numbers:

(1) $$\frac{1}{7\times8}+\frac{1}{8\times9}+\frac{1}{9\times10}+\frac{1}{10\times11}+\dots$$

In a sum like this, the finding of the L.C.M. and the multiplications, divisions, additions, cancellations etc., will be tiresome and disgusting. But our recognition of this series as coming under its right particular classification enables us to say at once: $S_4=\frac{4}{7\times11}=\frac{4}{77}$ and so on,

Note:—The principle explained above is in constant requisition in connection with the "Summation of Series" in Higher Algebra etc., and therefore of the utmost importance to the mathematician and the statistician, in general.

FIFTH TYPE

There is also a fifth type of fraction-addition$ (dealing with simple equations) which we often come across, which are connected with the "Summation of Series" (as in the previous type) and which we may readily tackle with the aid of the same (*Antyayoreva*) formula.

The characteristic peculiarity here is that each numerator is the difference between the two Binomial factors of its Denominator. Thus,

(1) $\frac{a-b}{(x+a)(x+b)}+\frac{b-c}{(x+b)(x+c)}+\frac{c-d}{(x+c)(x+d)}\ldots$

$\therefore S_3=\frac{a-d}{(x+a)(x+d)}$

Both the Algebraical explanations hereof are exactly as before (and need not be repeated here).

(2) $\frac{x-y}{(a+x)(a+y)}+\frac{y-z}{(a+y)(a+z)}+\frac{z-w}{(a+z)(a+w)}+\ldots$

$\therefore S_3=\frac{x-w}{(a+x)(a+w)}$

(3) $\frac{1}{(x+7)(x+8)}+\frac{2}{(x+8)(x+10)}+\frac{14}{(x+10)(x+24)}$

$\therefore S_3=\frac{17}{(x+7)(x+24)}$

(4) $\frac{3}{(x+7)(x+10)}+\frac{9}{(x+10)(x+19)}+\frac{27}{(x+19)(x+46)}$

$+\frac{99}{(x+46)(x+145)}+\ldots$

$\therefore S_4=\frac{138}{(x+7)(x+145)}$

(6) $\frac{a-b}{(px+a)(px+b)}+\frac{b-c}{(px+b)(px+c)}+\frac{c-d}{(px+c)(px+d)}$

$\therefore S_3=\frac{a-d}{(px+a)(px+d)}$

Note :— (i) If, instead of d, there be a in the last term (in this case), the Numerator in the answer becomes zero; and consequently the L.H.S. (i.e. the sum of the various fractions) is zero.

(ii) The difference between the Binomial factors of the Denominator in the L.H.S. is the Numerator of each fraction; and this characteristic will be found to charscterise the R.H.S. also.

(iii) The note at the end of the previous sub-section (re: the summation of series) holds good here too.

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CHAPTER XVII

QUADRATIC EQUATIONS

In the Vedic mathematics **Sūtras,** CALCULUS comes in at a very early stage. As it so happens that DIFFERENTIAL calculus is made use of in the Vedic **Sūtras** for breaking a quadratic equation down at **sight** into two simple equations of **the** first degree and as we now go on to our study of the Vedic Siitras bearing on Quadratic equations, we shall begin this chapter with a breif exposition of the calculus.

Being based on **basic** and fundamental first principles (relating to limiting values), they justifiably **come** into the picture at a very early stage. But these have been expounded and explained with **enormous** wealth of details covering not merely the **Sūtras** themselves but also the sub-siitras, axioms, corollaries, implications etc. We do not propose to go into the arguments by which the calculus has been established but shall content ourselves with an exposition of the rules enjoined therein and the actual *Modus Operandi.* The principal rules are briefly given below :

(i) In every quadratic expression (put in its standard form **i.e.** with 1 as the coefficient of x^a), the sum of its two Binomial **factors** is its first differential.

Thus, as regards the quadratic expression x^2-5x+6, we know its binomial factors are $(x-2)$ and $(x-3)$. And therefore, we can at once **say that** $(2x-5)$ (which is the sum of these two factors) is its D_1 (**i.e.** first DIFFERENTIAL).

(ii) This first differential **(of** each term) can **also** be obtained by multiplying its ध्वज (Dhwaja) **घात (Ghāta)** (**i.e.** the power by the अङ्क (**Aṅka** i.e. its coefficient) and **reducing** it by one.

Thus, as regards x^2-5x+6

x^2 gives $2x$; $-5x$ gives -5 ; and 6 **gives** zero.

$\therefore D_1 = 2x-5.$

(iii) Defining the DISCRIMINANT as the square of the coefficient of the middle term *minus* the product of double the first coefficientand double the independent term, the text then lays down the very important propositionthat the first differential is equal to the square root of the discriminant.

In the above case $x^2-5x+6=0$

$\therefore 2x-5 = \pm\sqrt{25-24} = \pm 1$

Thus the given quadratic equation is broken down at sight into the above two simple equations i.e. $2x-5=1$ and $2x-5=-1$ $\therefore$ $x=2$ or 3

The current modern method (dealing with its standard quadratic equation $ax^2+bx+c=0$) tells us that:

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ This is no doubt all right, so far as it goes; but it is still a very crude and clumsy way of stating that the first differential is the square root of the discriminant.

Another Indian method (of medieval times well-known as Shree Sbreedharacharya's method) is a bit better than the current modern Methods; but that too comes nowhere near the Vedic method which gives us (1) the relationship of the differential with the original quadratic (as the sum of its factors) and (2) its relationship with the discriminant as its square root! and thirdly, breaks the original quadratic equation-at sight-into two simple equations which immediately give us the two values of x!

A few more illustrations are shown hereunder:

(1) $4x^2-4x+1=(2x-1)(2x-1)=0$ $\therefore 8x-4=0$

(2) $7x^2-5x-2=(x-1)(7x+2)=0$ $\therefore 14x-5=\pm\sqrt{81}=\pm 9$

(3) $x^2-11x+10=(x-10)(x-1)=0$ $\therefore 2x-11=\pm\sqrt{81}=\pm 9$

(4) $6x^2+5x-3=0$ $\therefore 12x+5=\pm\sqrt{97}$

(5) $7x^2-9x-1=0$ $\therefore 14x-9=\pm\sqrt{109}$

(6) $5x^2-7x-5=0$ $\therefore 10x-7=\pm\sqrt{149}$

(7) $9x^2-13x-2=0$ $\therefore 18x-13=\pm\sqrt{241}$

(8) $11x^2+7x+7=0$ $\therefore 22x+7=\pm\sqrt{-259}$

(9) $ax^2+bx+c=0$ $\therefore 2ax+b=\pm\sqrt{b^2-4ac}$

This portion of the Vedic Siitras deals also with the Binomial theorem, factorisations, factorials, repeated factors, continued fractions. Differentiations, Integrations, Successive Differentiations, Integrations by means of continued fractions etc. But just now we are concerned only with the just hereinabove explained use of the differential calculus in the solution of quadratic equations (in general) because of the relationship $D_1 = \pm\sqrt{\text{the discriminant}}$. The other applications just referred to will be dealt with at later stages in the student's progress.

II This calculus-method is perfectly GENERAL i.e. it applies to *all* cases of quadratic equations. There are, however, certain special types of quadratic equations which can be still more easily and still more rapidly solved with the help of the special Sūtras applicable to them. Some of these formulas are old friends but in a new garb and a new set-up, a new context and so on. And they are so efficient in the facilitating of mathematics work and in reducing the burden of the toil therein. We therefore go on to some of the most important amongst these special types.

FIRST SPECIAL TYPE

(Reciprocals)

This deals with Reciprocals. The equations have, under the current system, to be worked upon laboriously, before they can be solved. For example:

(1) $x+\frac{1}{x}=\frac{17}{4}$

According to the current system, we say:

$\therefore \frac{x^2+1}{x}=\frac{17}{4}$

$\therefore 4x^2+4=17x$

$\therefore 4x^2-17x+4=0$

$\therefore (x-4)(4x-1)=0 \quad \therefore x=4 \text{ or } 1/4.$

or (ii) $\therefore x=\frac{17\pm\sqrt{289-64}}{8}=\frac{17\pm 15}{8}=4 \text{ or } 1/4$

But, according to the *Vilokanam* sub-Siitra of Vedic mathematics, we observe that the L.H.S. is the sum of two Reciprocals, split the $\frac{17}{4}$ of the R.H.S. into $4+\frac{1}{4}$ and at once say:

$x+\frac{1}{x}=4+\frac{1}{4}$ $\therefore x=4$ or $1/4$. It is a .matter of simple observation and no more.

(2) $x+\frac{1}{x}=\frac{26}{5}=5\frac{1}{5}$ $\therefore x=5$ or $\frac{1}{5}$

(3) $\frac{x}{x+1}+\frac{x+1}{x}=82/9=9\frac{1}{9}$ $\therefore \frac{x}{x+1}=9$ or $1/9$

(4) $\frac{x+1}{x+2}+\frac{x+2}{x+1}=\frac{37}{6}=6\frac{1}{6}$ $\therefore \frac{x+1}{x+2}=6$ or $\frac{1}{6}$

(5) $\frac{x+4}{x-4}+\frac{x-4}{x+4}=\frac{10}{3}=3\frac{1}{3}$ $\therefore \frac{x+4}{x-4}=3$ or $\frac{1}{3}$

(6) $x+\frac{1}{x}=\frac{13}{6}$

Here the R.H.S. does not readily seem to be of the same sort as the previous examples. But a little observation will suffice to show that $\frac{13}{6}$ can be split up into $\frac{3}{2}+\frac{2}{3}$

$\therefore x+\frac{1}{x}=\frac{3}{2}+\frac{2}{3}$ $\therefore x=\frac{3}{2}$ or $\frac{2}{3}$

(7) $x+\frac{1}{x}=\frac{25}{12}=\frac{4}{3}+\frac{3}{4}$ $\therefore x=\frac{4}{3}$ or $\frac{3}{4}$

(8) $\frac{x+5}{x+6}+\frac{x+6}{x+5}=\frac{29}{10}=\frac{5}{2}+\frac{2}{5}$ $\therefore \frac{x+5}{x+6}=\frac{5}{2}$ or $\frac{2}{5}$

(9) $\frac{x}{x+1}+\frac{x+1}{x}=\frac{169}{60}=\frac{12}{5}+\frac{5}{12}$ $\therefore \frac{x}{x+1}=\frac{12}{5}$ or $\frac{5}{12}$

(10) $\frac{2x+11}{2x-11}+\frac{2x-11}{2x+11}=\frac{193}{84}=\frac{12}{7}+\frac{7}{12}$ $\therefore \frac{2x+11}{2x-11}=\frac{12}{7}$ or $\frac{7}{12}$

(11) $x-\frac{1}{x}=\frac{5}{6}$

Here the connecting symbol is a *minus*. Accordingly, we say: $x-\frac{1}{x}=\frac{3}{2}-\frac{2}{3}$ $\therefore x=\frac{3}{2}$ or $-\frac{2}{3}$

M.B. :—Note the *minus* (of **the** second root) very carefully. For, the value $x=\frac{2}{3}$ will give us not $\frac{5}{6}$ but $-\frac{5}{6}$ on the R.H.S. and will therefore be wrong !

(12) $x-\frac{1}{x}=\frac{45}{14}=\frac{7}{2}-\frac{2}{7}$ $\therefore x=\frac{7}{2}$ or $-\frac{2}{7}$

(13) $\frac{x}{x+3}-\frac{x+3}{x}=\frac{15}{56}=\frac{8}{7}-\frac{7}{8}$ $\therefore \frac{x}{x+3}=\frac{8}{7}$ or $-\frac{7}{8}$

(14) $\frac{x+7}{x+9}-\frac{x+9}{x+7}=\frac{32}{63}=\frac{9}{7}-\frac{7}{9}$ $\therefore \frac{x+7}{x+9}=\frac{9}{7}$ or $-\frac{7}{9}$

(15) $\frac{5x+9}{5x-9}-\frac{5x-9}{5x+9}=\frac{56}{45}=\frac{9}{5}-\frac{5}{9}$ $\therefore \frac{5x+9}{5x-9}=\frac{9}{5}$ or $-\frac{5}{9}$

Note :—In the above examples, the L.H.S. was of the form $\frac{a}{b}\pm\frac{b}{a}$; and, consequently, we had to split the R.H.S. into the same form $\left(\frac{a}{b}\pm\frac{b}{a}\right)$

And this, when simplified, $=\frac{a^2\pm b^2}{ab}$

In other words, the Denominator on the R.H.S. had to he factorised into two factors, the sum of whose squares (or their difference, as the case may be) is the Numerator.

As this factorisation and the addition or subtraction of the squares will not always be easy and readily possible, we shall, at a later stage, expound certain rules which will facilitate this work of expressing a given number as the sum of two squares or as the difference of two squares.

SECOND SPECIAL TYPE.

(Under the *Śūnyaṃ Samuccaye* Formula.)

We now take up a second special type of quadratic equations which a very old friend (the *Śūnyaṃ Samuccaye* Sūtra) can help us to solve, at sight (a sort of problem which the mathematicians all regard as "Hard") !

We may first remind the student of that portion of an earlier chapter wherein, referring to various applications of the *Samuccaye Sūtra*, we dealt with the easy method by which the oneness of the sum of the numerator on the one hand and the

denominator on the other gave us one root and the oneness of the difference between the Numerator and the Denominator (on both sides) gave us another root, of the same Quadratic equation. We need not repeat all of it but only refer back to that portion of this volume and remind the student of the kind of illustrative examples with which we illustrated our theme :

(1) $\frac{3x+4}{6x+7}=\frac{5x+6}{2x+3}$ $\therefore 8x+10=0$; or $3x+3=0$

(2) $\frac{7x+5}{9x-5}=\frac{9x+7}{7x+17}$ $\therefore 16x+12=0$; or $2x-10=0$

(3) $\frac{7x-9}{2x-9}=\frac{9x-7}{14x-7}$ $\therefore 16x-16=0$; or $5x=0$

(4) $\frac{16x-3}{7x+7}=\frac{2x-15}{11x-25}$ $\therefore 18x-18=0$; or $9x-10=0$

THIRD SPECIAL TYPE

There is a third special type of Quadratic Equations which Is also generally considered "very hard" but whereof one root is readily yielded by the same ancient friend (the "*Sāmya Samuccaye*" *Sūtra* and the other is given by another friend-not so ancient, however but still quite an old friend i.e. the "*Śūnyaṃ Anyat*" *Sūtra* which was used for a special type of simultaneous equations.

Let us take a concrete instance of this type. Suppose we have to solve the equation:

$$\frac{2}{x+2}+\frac{3}{x+3}=\frac{4}{x+4}+\frac{1}{x+1}$$

The nature of the characteristics of this special type will be recognisable with the help of the usual old test and an additional new test.

The TESTS are: $\frac{2}{1}+\frac{3}{1}=\frac{4}{1}+\frac{1}{1}$; and $\frac{2}{2}+\frac{3}{3}=\frac{4}{4}+\frac{1}{1}$

In all such cases, "*Śūnyam Anyat*" formula declares that one root is zero ; and the "*Śūnyam Samuccaye*" Sūtra says : $D_1+D_2=0$ $\therefore 2x+5=0$ $\therefore x=-2\frac{1}{2}$

The Algebraical Proof hereof is as follows:

$\frac{2}{x+2}$ (by simple division) $=1-\frac{x}{x+2}$; and so on,

So, $\left\{1-\frac{x}{x+2}\right\}+\left\{1-\frac{x}{x+3}\right\}=\left\{1-\frac{x}{x+4}\right\}+\left\{1-\frac{x}{x-1}\right\}$

$\therefore$ (Removing $1-1$ and $1-1$ from both sides) x, (the common factor of all term) $=0$;

and on its removal, $\frac{1}{x+2}+\frac{1}{x+3}=\frac{1}{x+4}+\frac{1}{x+1}$

$\therefore$ (By the *Samuccaye* formula) $2x+5=0$;

Note:—In all these cases, *Vilokanam* (i.e. mere observation) gives us both the roots.

A few more illustrations of this special type are given:

(1) $\frac{3}{x+3}+\frac{4}{x+4}=\frac{2}{x+2}+\frac{5}{x+5}$ $\therefore x=0$ or $-3\frac{1}{2}$

(2) $\frac{1}{2x+1}+\frac{1}{3x+1}=\frac{2}{3x+2}+\frac{1}{6x+1}$

Now, $\because$ $\frac{1}{1+2x}=1-\frac{2x}{1+2x}$ and so on,

$\therefore \frac{2x}{2x+1}+\frac{3x}{3x+1}=\frac{3x}{3x+2}+\frac{6x}{6x+1}$

$\therefore x=0$

or (by cross-multiplication) $12x+5$ or $(36x+15)=0$

$\therefore x=-5/12$

(3) $\frac{a}{x+a}+\frac{b}{x+b}=\frac{a-c}{x+a-c}+\frac{b+c}{x+b+c}$

$\therefore x=0$ or $-\frac{1}{2}(a+b)$

(4) $\frac{a-}{x+a-b}+\frac{b-c}{x+b-c}=\frac{a+b}{x+a+b}-\frac{b+c}{x-b-c}$

$\therefore x=0$ or $\frac{1}{2}(c-a)$

(5) $\frac{a+b}{x+a+b}+\frac{b+c}{x+b+c}=\frac{2b}{x+2b}+\frac{a+c}{x+a+c}$

$\therefore x=0$ or $-\frac{1}{2}(a+2b+c)$

FOURTH SPECIAL TYPE

And again, there is still another special type of Quadratics which are "harder" but which our old friends "*Śūnyam Anyat*" and "*Parāvartya*" (Merger) can help us to solve easily.

Note:—Apropos of the subject-matter of the immediately preceding sub-section (the 3rd special type), let us now consider the equation $\frac{2}{x+2}+\frac{3}{x+3}=\frac{5}{x+5}$ This may look, at the outset, a like, but really is *not*, a quadratic equation of the type dealt with in the immediately previous sub-section (under *Śūnyam Anyat* and *Śūnyam Sāmya Samuccaye*) but only a simple MERGER (because, not only is the number of terms on the R.H.S. one short of the number required but also $\frac{2}{2}+\frac{3}{3}\neq\frac{5}{5}$ It is really a case under *Śūnyam Anyat* and *Parāvartya* (merger).

Here, the TEST is the usual one for the merger process i.e. N_1+N_2 (on the L.H.S.)$=N_3$ (on the R.H.S.) Thus:

$$\frac{2}{x+2}+\frac{3}{x+3}=\frac{5}{x+5}$$

$\therefore$ (By merger method) $\frac{-6}{x+2}-\frac{-6}{x+3}=0$ $\therefore 2x+5=0$

$\therefore x=-2\frac{1}{2}$

A few true illustrations are given below:

(1) $\frac{4}{x+2}+\frac{9}{x+3}=\frac{25}{x+5}$

Here $\because \frac{4}{2}+\frac{9}{3}=\frac{25}{5}$ $\therefore$ YES

$\therefore$ (By Division) $2-\frac{2x}{x+2}+3-\frac{3x}{x+3}=5-\frac{5x}{x+5}$

$\therefore x=0$. (This can be verified by mere observation)

or $\frac{2}{x+2}+\frac{3}{x+3}=\frac{5}{x+5}$ $\therefore$ (by merger), $x=-2\frac{1}{2}$

This result can be readily put down, by putting up *each numerator* over the absolute term of the Denominator as the Numerator of each term of the resultant equation and retaining the Denominator as before. [Or by taking the Square Root of Each Numerator] (in the present case).

Thus $\frac{4}{2}=2$; $\frac{9}{3}=3$; and $\frac{25}{5}=5$. And these will be our new Numerators. **Thus,** we have the newly derived Equation :

$$\frac{2}{x+2}+\frac{3}{x+3}=\frac{5}{x+5}$$

(By merger) $x=-2\frac{1}{2}$

(2) $\frac{2}{x+1}+\frac{9}{x+3}=\frac{25}{x+5}$

Here $\because \frac{2}{1}+\frac{9}{3}=\frac{25}{5}$ $\therefore$ **YES**

The derived equation is :

$$\frac{2}{x+1}+\frac{3}{x+3}=\frac{5}{x+5}$$

(which is the same as in all the three preceding cases).

$\therefore x=0$ or $-2\frac{1}{7}$

Note:—In the last two cases, the first term alone is different **and** yet, since the quotients $\frac{4}{2}$ and $\frac{2}{1}$ are the same, therefore it makes no difference to the result; and we get the same two roots in all the three cases!

(3) $\frac{6}{2x+3}+\frac{4}{3x+2}=\frac{4}{4x+1}$

Here $\because \frac{6}{3}+\frac{4}{2}=\frac{4}{1}$ $\therefore$ YES $\therefore x=0$

$\therefore$ or (By Division), $\frac{+4}{2x+3}-\frac{+6}{3x+2}-\frac{+16}{4x+1}$

N.B. Note that $\frac{2\times 6}{3}=4$, $\frac{3\times 4}{2}=6$ and $\frac{4\times 4}{1}=16$

and that these are the new Numerators (for the derived Equation)

$\therefore$ (By L.C.M.) $\frac{24}{12x+18}+\frac{24}{12x+8}=\frac{48}{12x+3}$: YES

or $\frac{1}{12x+18}+\frac{1}{12x+8}=\frac{2}{12x+3}$ $\therefore$ YES

... (By merger) $\frac{15}{12x+18}+\frac{5}{12x+8}=0$ ... $x=-\frac{7}{8}$

N.B.:—The **remaining examples in** this chapter **may** be held-over (if deemed advisable) for a **later reading**.

(4) $\frac{a}{x+a}+\frac{b}{x+b}=\frac{2c}{x+c}$

Here $\because \frac{a}{a}+\frac{b}{b}=\frac{2c}{c}$ $\therefore$ **YES** $\therefore x=0$

$\therefore \frac{1}{x+a}+\frac{1}{x+b}=\frac{2}{x+c}$ $\therefore$ **YES**

$\therefore$ (By merger) $\frac{a-c}{x+a}+\frac{b-c}{x+b}=0$

$\therefore x=\frac{bc+ca-2ab}{a+b-2c}$

(6) $\frac{a^2-b^2}{x+a+b}+\frac{b^a-c^2}{x+b+c}=\frac{a^2-c^2}{x+a+c}$

Here $\because \frac{a^2-b^2}{a+b}+\frac{b^2-c^2}{b f c}=\frac{a^2-c^2}{a+c}$ $\therefore$ **YES** $\therefore x=0$

OR $\frac{a-b}{x+a+b}+\frac{b-c}{x+b+c}=\frac{a-c}{x+a+c}$ $\therefore$ **YES**

$\therefore$ (By Merger) $\frac{(a-b)(b-c)}{(x+a+b)}+\frac{(b-c)(b-a)}{x+b+c}=0$ and so on.

(6) $\frac{1}{ax+d}+\frac{1}{bx+d}=\frac{2}{cx+d}$

Here (by division), we have : $d+ax)\ 1\ (\frac{1}{d}$

$$1+\frac{ax}{d}$$

$$\overline{\ \ -\frac{ax}{d}}$$

$\therefore \frac{a}{ax+d}+\frac{b}{bx+d}=\frac{2c}{cx+d}$

$\therefore \frac{abc}{abcx+bcd}+\frac{abc}{abcx+acd}-\frac{2abc}{abcx+abd}$ ... YES

$\therefore \frac{1}{D_1}+\frac{1}{D_2}=\frac{2}{D_3}$

$\therefore \frac{bcd\ \ -abd}{abcx+bcd}+\frac{acd\ \ -abd}{abcx+acd}=0$

$\therefore \frac{bd(c-a)}{bc(ax+d)}=\frac{ad(b-c)}{ac(bx+d)}$

$$\therefore \frac{c-a}{ax+d}=\frac{b-c}{bx+d}$$

$$\therefore x=\frac{ad+bd-2cd}{ac+bc-2ab}$$

(7) $$\frac{ax+2d}{ax+d}+\frac{bx+3d}{bx+d}=\frac{2cx+5d}{cx+d}$$

Here $\because \frac{2d}{d}+\frac{3d}{d}=\frac{5d}{d}=0 \therefore$ YES $\therefore x=0$

Or (by division)

$$\frac{1}{abcx+bcd}+\frac{2}{abcx+acd}=\frac{3}{abcx+abd}$$

$\therefore$ (By merger), $\frac{bcd-abd}{D1}+\frac{acd-abd}{D_2}=0$

$$\therefore \frac{bd(a-c)}{bd(ax+d)}+\frac{2ad(b-c)}{ac(bx+d)}=0$$

$$\therefore \frac{a-c}{ax+d}+\frac{2(b-c)}{(bx+d)}=0$$

$$\therefore x=\frac{ad+bd-2cd}{ac+bc-2ab}$$

OR (by mere division *Parāvartya*) at the very first step.

$$\therefore \frac{d}{ax+d}+\frac{2d}{bx+d}=\frac{3d}{cx+d}$$

(which is the same as No. 6 supra)

$$\therefore x=\frac{ad+bc-2cd}{ac+bc-2ab}$$

CONCLUDING LINKING NOTE

(On Quadratic Equations)

In addition to the above, there are several other special types of Quadratic Equations, for which the Vedic Sūtras have made adequate provision and also suggested several beautifully interesting devices and so forth. But these we shall go into and deal with, at a later stage.

Just at present, we address ourselves to our next appropriate subject for this introductory and illustrative Volume namely, the solution of cubic and Biquadratic Equations etc.

CHAPTER XVIII

CUBIC EQUATIONS

We solve cubic equations in various ways:

(i) with the aid of the *Parcivartya* Sūtra, the *Lopnna-Sthāpana* Sūtra, the पूरणापूरणाभ्यां formula (*Pūrana-Apūrnābhayām*) which means "by the completion or nun-completion" of the square, the cube, the fourth power etc.)

(ii) by the method of Argumentation and Factorisatiori (as explained in a previous chapter).

The Pūraṇa Method

The *Pūraṇa* method is well-known to the current system. In fact, the usually-in-vogue general formula $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ for the standard quadratic $(ax^2+bx+c=0)$ has been worked out by this very method. Thus,

$$ax^2+bx+c=0$$

$\therefore$ (Dividing by a), $x^2+\frac{bx}{a}+\frac{c}{a}=0$

$$\therefore x^2+\frac{bx}{a}=\frac{-c}{a}$$

$\therefore$ (completing the square on the L.H.S.)

$$\therefore x^2+\frac{bx}{a}+\frac{b^2}{4a^2}=-\frac{c}{a}+\frac{b^2}{4a^2}=\frac{b^2-4ac}{4a^2}$$

$$\therefore \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

$$\therefore x+\frac{b}{2a}=\frac{\pm\sqrt{b^2-4ac}}{2a}$$

$$\therefore x=\frac{-b}{2a}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

This method of "completing the square" is thus quite well-known to the present-day mathematicians, in connection with the solving of Quadratic Equations. But this is only a fragmentary and fractional application of the General Formula which (in

conjunction with the *Parāvartya,* the *Lopana-Sthāpana* etc., Siitras) is equally applicable to cubic, biquadratic and other higher-degree equations as well.

Completing the Cubic

With regard to cubic equations, we combine the *Parcivartya Sūtra* (as explained in the 'Division by *Parāvartya*' chapter) and the *Pūraṇa* sub-formula. Thus,

(1) $x^3-6x^2+11x-6=0$

$\therefore$ $x^3-6x^2=-11x+6$

But $(x-2)^3=x^3-6x^2+12x-8$

$\therefore$ (Substituting the value of x^a-6x^a from above herein), we have: $(x-2)^3=-11x+6+12x-8=x-2$

Let $x-2=y$ (and $\therefore$ let $x=y+2$)

$\therefore$ $y^3=y$ $\therefore$ $y=0$ or $\neq 1$ $\therefore$ $x=3$ or 1 or 2

N.B. :—It need hardly be poirted out that, by argumentation (re: the coefficients of x^3, x^2 etc.,) we can arrive at the same answer (as explained in a previous chapter dealing with factorisation by Argumentation) and that this holds good in all the cases dealt with in the present chapter.

(2) $x^3+6x^2+11x+6=0$

$\therefore$ $x^3+6x^2=-11x-6$

But $(x+2)^3=x^3+6x^2+12x+8=-11x-6+12x+8=x+2$

$\therefore$ $y^S=y$ (where y stands for $x+2$)

$\therefore$ $y=0$ or $\neq 1$ $\therefore$ $x=-2, -3$ or -1

(3) $x^3+6x^2-37x+30=0$

$\therefore$ $x^3+6x^2=37x-30$

$\therefore$ $(x+2)^3=x^3+6x^2+12x+8=49x-22=49(x+2)-120$

N.B. :—The object is to bring $(x+2)$ on the R.H.S. and thus help to formulate an equation in Y, obtain the three roots and then, by substitution of the value of x (in terms of y), obtain the three values of x.

$\therefore$ $y^3-49y+120=0$ $\therefore$ $(y-3)(y^2+3y-40)=0$

$\therefore$ $(y-3)(y-5)(y+8)=0$ $\therefore$ $y=3$ or 6 or -8

$\therefore$ $x=1$ or 3 or -10.

(4) $x^3+9x^2+23x+15=0$

$\therefore\ x^3+9x^2=-23x-15$

$\therefore\ (x+3)^3=(x^3+9x^2+27x-27)=4x+12=4(x+3)$

$\therefore\ y^3=4y\quad \therefore\ y=0 \text{ or } \pm 2\quad \therefore\ x=-3 \text{ or } -1 \text{ or } -5$

(5) $x^3+9x^2+24x+16=0$

$\therefore\ x^3+9x^2=-24x-16$

$\therefore\ (x+3)^3=(x^3+9x^2+27x+27)=3x+11=3(x+3)+2$

$\therefore\ y^3=3y+2\quad \therefore\ y^3-3y-2=0$

$\therefore\ (y+1)^2\ (y-2)=0\quad \therefore\ y=-1 \text{ or } 2\quad x=-4 \text{ or } -1$

(6) $x^3+7x^2+14x+8=0$

$\therefore\ x^3+7x^2=-14x-8$

$\therefore\ (x+3)^3=(x^3+9x^2+27x+27)=2x^2+13x+19=$
$(x+3)(2x+7)-2,$

$\therefore\ y^3=y(2y+1)-2\ \therefore y^3-y(2y+1)+2=0=(y-1)(y+1)$
$(y-2)$

$\therefore\ y=1 \text{ or } -1 \text{ or } 2\quad \therefore\ x=-2 \text{ or } -4 \text{ or } -1$

(7) $x^3+8x^2+17x+10=0\quad \therefore\ x^3+8x^2=-17x-10$

$\therefore\ (x+3)^3=(x^3+9x^2+27x+27)=x^2+10x+17=$
$(x+3)(x+7)-4$

$\therefore\ y^3=y(y+4)-4\quad \therefore\ y^3-y^2-4y-4=0\quad \therefore\ y=1 \text{ or } \neq 2$

$\therefore\ x=-2 \text{ or } -1 \text{ or } -5$

(8) $\therefore\ x^3+10x^2+27x+18=0$

Now $\therefore\ (x+4)^3=(x^3+12x^2+48x+64)$

Hence the L.H.S. $=(x+y)^3-(2x^2+21x+46)=(x+y)^3$
$\{(x+4)(2x+13)-6\}$

$\therefore\ y^3=y\ (2y+5)-6\quad \therefore\ (y-1)(y+2)(y-3)=0$

$\therefore\ y=1 \text{ or } -2 \text{ or } 3$

$\therefore\ x=-3 \text{ or } -6 \text{ or } -1$

Note:—Expressions of the form x^3-7x+6 can be split into $x^3-1-7x+7$ etc., and readily factorised. This is always applicable to all such cases (where x^2 is absent) and should be fully utilised.

The *Pūraṇa* method explained in this chapter for the solution of cubic equations will be found of great help in factorisation ;and vice-versa.

"Harder" cubic equations will be taken up later.

CHAPTER XIX

BIQUADRATIC EQUATIONS

The procedures (*Pūraṇa* etc.,) expounded in tho previous chapter for the solution of cubic equations can be equally well applied in the case of **Biquadratics** etc., too. Thus,

(1) $x^4+4x^3-25x^2-16x+84=0$

$\therefore\ x^4+4x^3=25x^2+16x-84$

$\therefore\ (x+1)^4=x^4+4x^3+6x^2+4x+1$

$=(25x^2+16x-84)+(6x^2+4x+1)=31x^2+20x-83$

$=(x+1)(31x-11)-72$

$\therefore\ y^4=y(31y-42)-72 \quad \therefore\ y^4-31y^2+42y+72=0$

$\therefore\ y=-1,\ 3,\ 4$ or -6

$\therefore\ x=-2,\ 2,\ 3$ or -7

(2) $x^4+8x^3+14x^2-8x-15=0$

$\therefore\ x^4+8x^3=14x^2+8x+15$

$\therefore\ (x+2)^4=x^4+8x^3+24x^2+32x+16=10x^2+4x+31$

$=(x+2)(10x+20)-9=10(x+2)^2-9$

$\therefore\ y^4=10y^2-9 \quad \therefore\ y^2=1$ or $9 \quad \therefore\ y=\pm 1$ or ± 3

$\therefore\ x=-1$ or -3 or 1 or -5

(3) $x^4-12x^3+49x^2-78x+40=0$

$\therefore\ x^4-12x^3=-49x^2+78x-40$

$\therefore\ (x-3)^4=x^4-12x^3+54x^2-108x+81$

$=5x^2-30x+41=(x-3)(5x-15)-4=5(x-3)^2-4$

$\therefore\ y^4-5y^2+4=0 \quad \therefore\ y^2=1$ or $4 \quad \therefore\ y=\pm 1$ or ± 2

$\therefore\ x=4$ or 5 or 2 or 1

(4) $x^4+16x^3+86x^2+176x+105$

$\therefore\ x^4+16x^3=-86x^2-176x-105$

$\therefore\ (x+4)^4=x^4+16x^3+96x^2+256x+256$

$=10x^2+80x+151=(x+4)(10x+40)-9$

$=10(x+4)^2-9$

$\therefore\ y^4-10y^2+9=0 \quad \therefore\ y^2=1$ or $9 \quad \therefore\ y=\pm 1$ or ± 3

$\therefore\ x=-3$ or -5 or -1 or -7

(5) $x^4-16x^3+91x^2-216x+180=0$

$\therefore x^4-16x^3=-91x^2+216x-180$

$\therefore (x-4)^4=x^4-16x^3+96x^2-256x+256$

$=5x^2-40x+76=(x-4)(5x-20)-4=5\ (x-4)^2-4$

$\therefore y^4-5y^2+4=0\quad \therefore y^2=1$ or $4\quad \therefore y=\neq 1$ or $\neq 2,$

$\therefore x=3$ or 5 or 6 or 2

(6) $x^4-20x^3+137x^2-382x+360=0$

$\therefore x^4-20x^3=-137x^2+382x-360$

$\therefore (x-5)^4=x^4-20x^3+150x^2-500x+625$

$=13x^2-118x+265=(x-5)(13x-53)$

$\therefore y^4=y(13y+12)\quad \therefore y=0$ or $y^3-13y-12=0$

$\therefore y=0$ or $(y+1)\ (y+3)(y-4)=0$

$\therefore y=0$ or -1 or -3 or 4

$\therefore x=5,\ 4,\ 2$ or $9.$

Note:—The student need hardly be reminded that all these examples (which have all been solved by the *Pūraṇa* method hereinabove) can also be solved by the Argumentation-cam-factorisation method.

A SPECIAL TYPE

There are several special types of Biquadratic equations dealt with in the Vedic Sūtras. But we shall here deal with only one such special type and hold the others over to a later stage.

This type is one wherein the L.H.S. consists of the sum of the fourth powers of two Binomials (and the R.H.S. gives us the equivalent thereof in the shape of an arithmetical number.) The formula applicable to such cases is the व्यष्टि-समष्टि (*Vyaṣṭi Samaṣṭi*) Sūtra (or the Lopana Sthdpana one) which teaches us how to use the average or the exact middle binomial for breaking the Biquadratic down into a simple quadratic (by the easy device of mutual cancellation of the old powers i.e. the x^3 and the x).

A single concrete illustration will suffice for explaining this process :

$(x+7)^4+(x+5)^4=706.$

Let $x+6$ (the average of the two Binomials) $=a$

$\therefore (a+1)^4+(a-1)^4=706$

$\therefore$ owing to the cancellation of the odd powers x^3 and x,

$2a^4+12a^2+2=706 \quad \therefore a^4+6a^2-352=0$

$\therefore a^2=16$ or $-22 \quad \therefore a=\pm 4$ or $\pm\sqrt{-22}$

$\therefore x=-2$ or -10 or $\pm\sqrt{-22}-6$

N.B. :—In simple examples like this, the integral roots are small ones and can be spotted out by mere inspection and the splitting up of 706 into 625 and 81 and for this purpose, the *Vilokanam* method will suffice. But, in cases involving more complex numbers, fractions, surds, imaginary quantities etc., and literal coefficients and so on : *Vilokanam* will not completely solve the Equation. But here too, the *Vyaṣṭi-Samaṣṭi* formula will quite serve the purpose. Thus,

The General *Formula* will be as follows:

Given $(\overline{x+m}+n)^4+(\overline{x+m}-n)^4=p$

$$\therefore a^4+6a^2+\left(n^4-\frac{p}{2}\right)=0$$

$$\therefore a^2=\frac{-6f\sqrt{36-4n^4+2p}}{2}$$

$$\therefore a=\pm\sqrt{\frac{-6\pm\sqrt{36-4n^4+2p}}{2}}$$

$$\therefore x=-m\pm\sqrt{\frac{-6\pm\sqrt{36-4n^4+2p}}{2}}$$

Applying this to the above example, we have :

$$x=-6\pm\sqrt{\frac{-6\pm\sqrt{1444}}{2}}=-6\pm\sqrt{\frac{-6\pm 38}{2}}$$

$$=-6\pm\sqrt{16} \text{ or } \sqrt{-22}=-6(\pm 4 \text{ or } \pm\sqrt{-22})-6$$

(which tallies with the aobve)

N.B. :— "Harder" Biqudratics, Pentics etc., will be taken up later.

CHAPTER XX

MULTIPLE SIMULTANEOUS EQUATIONS.

We now go on to the solution of Simultaneous Equations involving three (or more) unknowns. The ***Lopam-Sthipanu*** Sūtra, the ***Ānurūpya*** Sūtra and the ***Parāvartya*** Sūtra are the ones that we make use of for this purpose.

FIRST TYPE

In the first type thereof, we have a significant figure on the R.H.S. in only one equation (and zeroes in the other two) From the homogeneous zero equations, we derive new equations defining two of the unknowns in terms of the third; we then substitute these values in the third equation ; and thus we obtain the values of all the three unknowns.

A second method is the judicious addition and subtraction of proportionate multiples for bringing about the elimination of one unknown and the retention of the other two.

In both these methods, we oan make our own choice of the unknown to be eliminated. the multiples to be taken etc., Thus :

(1) $x+y-z=0$ (A)
$4x-5y+2z=0$ (B)
$3x+2y+z=10$ (C)

(i) A+C gives us: $4x+3y=10$; & 2A+B gives us: $6x-3y=0$ $\}$ $\therefore 10x=10$
$\therefore x=1$, $y=2$ and $z=3$

(ii) from A, we have $x+y=z$ and from B, we have $4x-5y=-2z$

$\therefore$ By *Parāvartya*, $x=\frac{1}{3}z$; and $y=\frac{2}{3}z$

$\therefore$ (by substitution in b $z=1\frac{1}{3}z+z=10$ $\therefore z=3$, $x=1$ and $y=2$

(2) $7y-11z-2x=0 \quad \ldots \quad A$
$8y-\ z-6x=0 \quad \ldots \quad B$
$3x+4y+5z=35 \quad \ldots \quad C$

(i) Adding B and 2C, we have $16y+3z=70$
Subtracting B from 3A, $13y-26z=0$

$\therefore y=\frac{1820}{455}=4$; and $z=\frac{910}{455}=2$

$\therefore$ (Substituting these values in C) $\therefore x=3$

(ii) $\therefore 7y-11z=2x$
$8y-7z=6x$
$\therefore y=\frac{-52x}{-39}=1\frac{1}{3}x$;
and $z=\frac{-26x}{-39}=\frac{2}{3}x$

$\therefore 3x+5\frac{1}{3}x+3\frac{1}{3}x=35 \quad \therefore x=3$
$y=4$
and $z=2$

(3) $2x-3y+4z=\ 0 \quad \ldots \quad \ldots \quad (A)$
$7x+2y-6z=\ 0 \quad \ldots \quad \ldots \quad (B)$
$4x+3y+\ z=37 \quad \ldots \quad \ldots \quad (C)$

(i) A+C gives us: $6x+\ 5z=37$
2A+3B gives us: $25x-10z=\ 0$
$\therefore x=\frac{370}{185}=2$; and $z=\frac{925}{185}=5$; and $y=8$

From (A) and (B) we have

(ii) $\therefore -3y+4z=-2x$
$2y-6z=-7x$ $\ldots \ y=\frac{-40x}{-10}=4x$; and $z=\frac{-25x}{-10}$
$=2\frac{1}{2}x$

$\therefore 4x+12x+2\frac{1}{2}x=37 \quad \therefore x=2$; $y=8$; and $z=5$

SECOND TYPE

This is one wherein the R.H.S. contains significant figures in all the three equations. This can be solved by *Parāvartya* (CROSS-multiplication) so as to produce two derived equations whose R. H. S. consists of zero only, or by the first or the second of the methods utilised in the previous sub-section. Thus,

$2x-\ 4y+\ 9z\ =28 \quad \ldots \quad \ldots \quad A$
$7x+\ 3y-\ 5x\ =\ 3 \quad \ldots \quad \ldots \quad B$
$9x+10y-11z\ =\ 4 \quad \ldots \quad \ldots \quad C$

(1) (i) $\therefore$ $196x+84y-140z=84$ and $6x-12y+27z=84$ $\}$ $\therefore$ $190x+96y+167z=0$

$\therefore$ $28x+12y-20z=12$ and $27x+30y-33z=12$ $\}$ $\therefore$ $x-18y+13z=0$

Having thus derived two equations of this kind (i.e. of the first special type), we can now follow the first method under that type; and, after a lot of big multiplications, subtractions, and divisions, we can obtain the answer: $x=2$, $y=3$ and $z=4$

(ii) or, (adopting the first method adopted in the last sub-section), we have:

$\therefore$ $3y-5z=3-7x$ and $10y-11z=4-9x$

$\therefore$ By CROSS-multiplication,

$$y=\frac{-20+45x+33-77x}{-17}=\frac{32x-13}{17};$$

$$\text{and } z=\frac{30-70x-12+27x}{-17}=\frac{43x-18}{17}$$

$$\therefore\ 2x-\frac{128x-52}{17}+\frac{387x-162}{17}=28$$

$\therefore$ $34x-128x+52+387x-162=476$ $\therefore$ $293x=586$

$\therefore$ $x=2$; $y=3$; and $z=4$

This method too involves a lot of clumsy labour.

(iii) or, (adopting the *Lopana-Sthāpana* method), we say:

C−A−B gives us $11y-15z=-27$ and $9B-7C$ gives us $-43y+32z=-1$ $\}$

$\therefore$ $y=3$, $z=4$ and $x=2$

(2) $x+2y+3z=12$ ... A
$2x+3y+4z=18$ ... B
$4x+3y+5z=24$ ... C

(i) $\therefore$ $24x+36y+48z=216$, $18x+36y+54z=216$ $\}$ $\therefore$ $6x-6z=0$ $\therefore$ $x-z=0$

Similarly $48x+36y+60z=288$, $24x+48y+72z=288$ $\}$ $\therefore$ $24x-12y-12z=0$ $\therefore$ $2x-y-z=0$

$\therefore$ $x=y=z=2$

(ii) $\therefore$ $2y+3z=12-x$; and $3y+4z=18-2x$ $\therefore$ $y=6-2x$; and $z=x$ $\therefore$ $x=y=z=2$.

(iii) 2A−B gives us: $y+2z=6$ } $\therefore y=2$ }
2B−C gives us: $3y+3z=12$ } and $z=2$, $x=2$ }

or (iv) by mere observation.

(3) $x+2y+3z=14$ ... A
$2x+3y+4z=20$ ... B
$3x+y+6z=23$ ... C

$\therefore$ $28x+42y+56z=280$
$20x+40y+60z=280$ | $\therefore$ $8x+2y-4z=0$

and $42x+14y+84z=322$
$23x+46y+69z=322$ | $\therefore$ $19x-32y+15z=0$

and so on as before.

(ii) $\therefore$ $2y+3z=14-x$ } $\therefore$ $y=60-8x-58+4x=4-2x$
$3y+4z=20-2x$ } $z=42-3x-40+4x=x+2$

$\therefore$ $3x+4-2x+6x+12=23$ $\therefore$ $x=1$, $y=2$ and $z=3$

(iii) 2A−B gives us: $y+2z=8$ } $\therefore$
and 3A−C gives us; $5y+3z=19$ }
$x=1$

(4) $x+2y+3z=11$ ... A
$2x+3y+4z=16$ ... B
$3x+5y+6z=25$ ... C

(i) $(16x+32y+48z)-(22x+33y+44z)=-6x-y+4z=0$
and $(33x+55y+66z)-(25x+50y+75z)=8x+5y-9z=0$

and so on.

(ii) $\therefore$ $2y+3z=11-x$ } and so on, as before.
$3y+4z=16-2x$ }

(iii) $x+y+z=5$ } $\therefore$ $y+z=4$;and $x+z=3$
and $x+2y+2z=9$ } $\therefore$ $x=1$, $y=2$ and $z=2$

In all these processes, there is an element, more or less, of clumsiness and cumbrousness which renders them unfit to come under and fit satisfactorily into the Vedic category. Methods expounded in the Vedic Sūtras and free from the said draw-back and also capable of universal application will be explained st a later stage.

CHAPTER XXI

SIMULTANEOUS QUADRATIC EQUATIONS

The **Sūtras** needed for the **solution** of simultaneous **Quadratic** equations have practically all been explained already. **Only** the actual applicational procedure, devices and *modus-operandi* thereof have to be explained. Thus:

(1) $\left.\begin{array}{l} x+y=5 \\ \& \ xy=6 \end{array}\right\} \therefore \left.\begin{array}{l} x^2+2xy+y^2=25 \\ 4xy=24 \end{array}\right\} \begin{array}{l} \therefore (x-y)^2=1 \\ \therefore x-y=\pm 1 \end{array}$

$\therefore \left.\begin{array}{l} x=2 \\ 6=3 \end{array}\right\} or \left.\begin{array}{l} x=3 \\ y=2 \end{array}\right\}$

This is readily obtainable by ***Vilokanam*** (mere observation) and also because symmetrical values can always be reversed.

(2) $\left.\begin{array}{l} x-y=1 \\ \text{and } xy=6 \end{array}\right\} \therefore \left.\begin{array}{l} x=3 \\ y=2 \end{array}\right\} or \left.\begin{array}{l} -2 \\ -3 \end{array}\right\}$ ***Note*** the ***minus***

(3) $\left.\begin{array}{l} 5x-y=17 \\ \text{and } xy=12 \end{array}\right\} \therefore \left.\begin{array}{l} 25x^2-10xy+y^2=289 \\ \text{and } 20xy=240 \end{array}\right\} \begin{array}{l} \therefore (5x+y)^2=529 \\ \therefore 5x+y=\pm 23 \end{array}$

$\therefore 10x=40$ or -6 $\therefore \left.\begin{array}{l} x=4 \\ y=3 \end{array}\right\} or \left.\begin{array}{l} -\frac{3}{5} \\ -20 \end{array}\right\}$

N.B. :—1. When the value of x or y has been found, xy at once gives us the value of the other. Thus, if, here, $x=4$, $y=3$, no other substitution etc., is necessary.

2 One set of values can be found out by ***Vilokanam*** alone.

3. The internal relationship between the two sets of Values will be explained later.

(4) $\left.\begin{array}{l} 4x-3y=7 \\ \text{and } xy=12 \end{array}\right\}$ $\therefore x=4$ and $y=3$ by (mere *Vilokanam* observation)

(ii) $(4x-3y)^2=49$ $\therefore 4x+3y=\pm 25$ $\therefore 8x=32$ or -18

$\therefore \left.\begin{array}{l} x=4 \\ y=3 \end{array}\right\} or \left.\begin{array}{l} -\frac{9}{4} \\ -5\frac{1}{3} \end{array}\right\}$

(5) $\left.\begin{array}{l} x^3-y^3=19 \\ x-y=1 \end{array}\right\} \therefore \left.\begin{array}{l} x^2+xy+y^2=19 \\ \text{and } x^2-2xy+y^2=1 \end{array}\right\} \therefore 3xy=18 \therefore xy=6$

$\therefore \left.\begin{array}{l} x=3 \\ y=2 \end{array}\right\} or \left.\begin{array}{l} -2 \\ -3 \end{array}\right\}$

(6) $\left.\begin{array}{l}x^3+y^3=61\\ x+y=1\end{array}\right\}\therefore \left.\begin{array}{l}x^2-xy+y^2=61\\ x^2+2xy+y^2=1\end{array}\right\}\therefore 3xy=-60 \therefore xy=-20$ $\therefore \left.\begin{array}{l}x=5\\ y=-4\end{array}\right\}$ *or* $\left.\begin{array}{r}-4\\ 5\end{array}\right\}$

N.B. :—There is *plus* sign all through. Therefore it can all be simply reversed (i.e. one by ***Vilokanam*** and the other by reversal).

(7) $\left.\begin{array}{l}x+y=4\\ \text{and } x^2+xy+4x=24\end{array}\right\}$ $\therefore$ (i)By ***Vilokanam***, $x=3$ and $y=1$

Secondly $x(x+y)+4x=8x=24$ $\therefore \left.\begin{array}{l}x=3\\ y=1\end{array}\right\}$

(8) $\left.\begin{array}{l}x+2y=5\\ \text{and } x^2+3xy+2y^2+4x-1=10\end{array}\right\}$

$\therefore (x+y)(x+2y)+4x-1=5x+5y+4x-y=10$

$\left.\begin{array}{l}\therefore 9x+4y=10\\ \text{But } x+2y=5\end{array}\right\}$

$\therefore$ (By ***Parāvartya*** or by ***Sūnyaṃ Anyat***) $x=0$ & $y=2\frac{1}{2}$

(9) $x+2y=5$

and $x^2+3xy-2y^2+4x+3y=0$

$\therefore (x+2y)(x+y)-4y^2+4x+3y=0$

$\therefore 5x+5y-4y^2+4x+3y=9x+8y-4y^2=0$

$\therefore 4y^2+10y-45=0 \quad \therefore 8y+10=\pm\sqrt{820}$

$\therefore y=\pm\frac{\sqrt{820}-10}{8}=\frac{-5\mp\sqrt{205}}{4}$

$\therefore x=\frac{15\mp\sqrt{205}}{2}$

(10) $\left.\begin{array}{l}x+y=5\\ 3x^2+y^2=19\end{array}\right\}$ $\therefore 3x^2+y^2=(x+y)(3x-3y)+4y$
$=15x-15y+4y^2=19$

$\therefore 75-15y-15y+4y^2=19$

$\therefore 4y^2-30y+56=0$

$\therefore 8y-30=\pm\sqrt{4}=\pm2 \quad \therefore 8y=32 \text{ or } 28 \quad \therefore \left.\begin{array}{l}y=4 \text{ or } 3\frac{1}{2}\\ \text{and } x=1 \text{ or } 1\frac{1}{2}\end{array}\right\}$

(11) $\left.\begin{array}{l}x^2+3x-2y=4\\ 2x^2-5x+3y=-2\end{array}\right\}\begin{array}{l}\therefore 7x^2-x-8=0\\ \therefore (x+1)(7x-8)=0\end{array}\left.\right\}\therefore \left.\begin{array}{l}x=-1\\ y=-3\end{array}\right\}$ *or* $\begin{array}{l}8/7\\ 18/49\end{array}$

(12) $\left.\begin{array}{l}x+y=5\\ x^2-2y^2=1\end{array}\right\}\begin{array}{l}\therefore 5x-5y-y^2=0\\ \therefore y^2+10y-24=0\end{array}$ $\begin{array}{l}\therefore 25-10y-y^2=1\\ \therefore \left.\begin{array}{l}y=2\\ x=3\end{array}\right\} \begin{array}{l}\text{or } -12\\ \text{or } -17\end{array}\end{array}$

(13) $\left.\begin{array}{l} 2x+\ \ y=3 \\ x^2+2xy=3 \end{array}\right\}$ $\therefore \left.\begin{array}{l} 1\frac{1}{2}x+2\frac{1}{4}y-\frac{3}{4}y^2=\ 3 \\ 6x+\ 9y-3y^2=12 \end{array}\right\}$ $\therefore \begin{array}{l} y^2-2y+1=0 \\ \therefore y=1 \text{ and } x=1 \end{array}$

or (ii) $\left.\begin{array}{l} 4x^2+2xy=6x \\ x^2+2ry=3 \end{array}\right\}$. $3x^2-6x+3=0$. $\left.\begin{array}{l} x=1 \\ \text{and } y=1 \end{array}\right\}$

(14) $\left.\begin{array}{l} x+y=2 \\ x^2+y^2+2x+3y=7 \end{array}\right\}$ $\begin{array}{l} \therefore 4x+y+2y^2=7 \quad \therefore 8-3y+2y^2=7 \\ \therefore 2y^2-3y+1=0 \quad \therefore \left.\begin{array}{l} y=1 \\ x=1 \end{array}\right\} \textit{ or } \left.\begin{array}{l} \frac{1}{2} \\ 1\frac{1}{2} \end{array}\right\} \end{array}$

(15) $\left.\begin{array}{l} 2x^2+xy+\ y^2=\ 8 \\ 3x^2-xy+4y^2=17 \end{array}\right\}$ $\therefore x^2+y^2=5$

And (by CROSS—multiplication)

$34x^2+17xy+17y^2=24x^2-8xy+32y^2$

$\therefore 10x^2+25xy-15y^2=0 \quad \therefore 2x^2+5xy-3y^2=0$

$\therefore (x+3y)\ (2x-y)=0 \quad \therefore x=-3y$ or fy

Substituting in $x^2+y^2=5$, we have

$9y^e+y^2=5$ or $\frac{5}{4}y^2=5 \quad \therefore y^2=\frac{1}{2}$ or 4

$\therefore y=\pm\frac{1}{\sqrt{2}}$ or $\neq 2$

and $\therefore x=\pm 3\sqrt{\frac{1}{2}}$ or $\pm\frac{1}{2}\sqrt{\frac{1}{2}}$ or $\neq 6$ or $\neq 1$.

N.B.:—Test for the correct sign (plus or minus).

(16) $\left.\begin{array}{l} 2x^2+\ xy+y^2=77 \\ 2x^2+3xy\qquad =92 \end{array}\right\}$ $\begin{array}{l} \therefore 184x^2+\ 92xy+92y^2=154x^2+231xy \\ .\quad 30x^2-139xy+92y^2=0 \end{array}$

$\therefore (5x-4y)\ (6x-23y)=0 \quad \therefore x=\frac{4}{5}y$ or $\frac{23}{6}y$

$\therefore$ (By substitution),

$\left.\begin{array}{l} y=\pm 5 \\ x=\pm 4 \end{array}\right\}$ or $\begin{array}{l} 2\pm\sqrt{6/7} \\ \frac{23}{3}\pm\sqrt{7} \end{array}$

(17) $\left.\begin{array}{l} 3x^2-4xy+2y^2=1 \\ y^2-x^2\qquad =-15 \end{array}\right\}$ $\begin{array}{l} \therefore \text{(By subtraction)}, 4x^2-4xy+2y^2=16 \\ \therefore 2x-y=\pm 4 \end{array}$

$\therefore$ (By substitution), $4x^2\mp 16x+16-x^2=-15$

$\therefore 3x^2\mp 16x+31=0$ & so on.

(18) $\left.\begin{array}{l} 2x^2-7xy+3y^2=\ 0 \\ x^2+xy+\ y^2=13 \end{array}\right\}$ $\begin{array}{l} \therefore x=3y \text{ or } \frac{1}{2}y \\ \therefore \\ \text{and } x=\pm 3 \end{array}$ $\Bigg\}$ or $\left.\begin{array}{l} \pm\sqrt{\frac{13}{7}} \\ \pm 2\sqrt{\frac{13}{7}} \end{array}\right\}$

(19) $\left.\begin{array}{l} 3x^2-4xy+2y^2=1 \\ y^2-x^2\qquad =0 \end{array}\right\}$ $\begin{array}{l} \therefore x=\pm y. \\ \therefore 3x^2-4x^2+2x^2=1 \quad \therefore \left.\begin{array}{l} x=\pm 1 \\ y=\pm 1 \end{array}\right\} \end{array}$

or $3x^2+4y^2+2y^2=1 \quad \therefore \left.\begin{array}{l} y=\pm\sqrt{1/3} \\ \text{and } x=\pm\sqrt{1/3} \end{array}\right\}$

(20) $\left.\begin{array}{l} x^2-xy=12y^2 \\ x^2+y^2=68 \end{array}\right\}$ $\therefore$ $x=4y$ or $-3y$

$\therefore$ By substitution, $17y^2=68$ or $10y^2=68$

$\therefore$ $y=\pm\sqrt{2}$ or $\pm\sqrt{34/5}$

and $x=\pm\sqrt{8}$ or $\pm 3\ \sqrt{34/5}$

(21) $\left.\begin{array}{l} x^2-2xy+y^2=2x-2y+3 \\ x^2+xy+2y^2=2x-\ y+3 \end{array}\right\}$

(i) By *Śūnyaṃ Anyat* $\therefore$ $y=0$

Let $x-y=a$ $\therefore$ $a^2-2a-3=0$ $\therefore$ $a=3$ or -1

$\therefore$ $x-y=3$ or ± 1.

Now, substitute and solve.

or (ii) By subtraction, $3xy+y^2=y$

$\therefore$ $y=0$ or $3x+y=1$

Substitute and solve

N.B. :—The *Śūnyaṃ Anyat* method is the best.

(22) $\left.\begin{array}{l} 3x^2+2xy-y^2=0 \\ x^2+y^2=2x(y+2x) \end{array}\right\}$ $\therefore$ $x=-y$ or $\frac{1}{3}y$

$\therefore$ Substitute and solve

or (ii) By transposition,

$-3x^2-2xy+y^2=0$

This means that the two equations are not independent ; and therefore. any value may be given to y and a corresponding set of values will emerge for x !

("Harder" simultaneous Quadratics will be taken up at a later stage).

Chapter XXII

FACTORISATION AND DIFFERENTIAL CALCULUS

In this Chapter the relevant **Sūtras** (*Guṇaka-Samuccaya* etc.,) dealing with successive differentiations, covering Leibnity's theorem, Maclaurin's theorem, Taylor's theorem etc., and given a lot of other material which is yet to be studied and decided on by the great mathematicians of the present-day western world, is also given.

Without going into the more abstruse details connected herewith, we shall, for the time-being, content ourselves with a very brief sketch of the general and basic principles involved and a few pertinent sample-specimens by way of illustration.

The basic principle is, of course, elucidated by the very nomenclature (i.e. the *Guṇaka-Samuccaya*) which postulates that, if and when a quadratic expression is the product of the Binomials $(x+a)$ and $(x+b)$, its first differential is the sum of the said two factors and so on (as already explained in the chapter on quadratic equations).

It need hardly be pointed out that the well-known rule of differentiation of a product (i.e. that if $y=uv$, when u and v be the function of x, $\frac{dy}{dx}=v\frac{du}{dx}+u\frac{dv}{dx}$) and the *Guṇaka-Samuccaya* Sūtras denote, connote and imply the same mathematical truth.

Let us start with very simple instances :

(1) $x^2+3x+2=\underset{a}{(x+1)}\ \underset{b}{(x+2)}$

$\therefore$ D_1 (the first differential$=2x+3=(x+2)+(x+1)=\Sigma a$

(2) $x^3+6x^2+11x+6=\underset{a}{(x+1)}\ \underset{b}{(x+2)}\ \underset{c}{(x+3)}$

$\therefore\ D_1=3x^2+12x+11=(x^2+3x+2)+(x^2+5x+6)$
$+(x^2+4x+3)=ab+bc+ac=\Sigma ab.$

$\therefore\ D_2=6x+12=2(3x+6)=2(x+1)+(x+2)+(x+3)=$
$=2(a+b+c)=2\ \Sigma a=\underline{|2}\ \Sigma a$

(3) $x^4+10x^3+35x^2+50x+24=(x+1)(x+2)(x+3)(x+4)$

$\therefore D_1=4x^3+30x^2+70x+50=\Sigma abc$

$D_2=12x^2+60x+70=2, \Sigma ab=|\underline{2}. \quad \Sigma ab.$

$D_3=24x+60=6(4x+10)=|\underline{3}\ \Sigma a$

(4) $x^5+15x^4+71x^3+178x^2+214x+120$

$=(x+1)(x+2)(x+3)(x+4)(x+5)$

$\therefore D_1=5x^4+60x^3+213x^2+356x+214=\Sigma abcd$

$\therefore D_2=20x^3+180x^2+426x+356=|\underline{2}\ \Sigma abc$

$\therefore D_3=60x^2+360x+426=|\underline{3}\ Cab$

$\therefore D_4=120x+360=24(5x+15)=|\underline{4}\ \Sigma a$

(5) $x^4+19x^3+234x^2+284x+240=(x+2)(x+3)(x+4)(x+10)$

$\therefore D_1=4x^3+57x^2+468x+284=\Sigma abc$

$\therefore D_2=12x^2+114x+468=|\underline{2}\ \Sigma ab$

$\therefore D_3=24x+114=6(4x+19)=|\underline{3}\ \Sigma a$

These examples will suffice to show the internal relationship subsisting between the factors of a Polynomial and the successive differentials of that Polynomial; and to show how easily, on knowing the former, we can derive the latter and vice versa.

There is another relationship too in another direction wherein factorisation and differentiation are closely connected with each other and wherein this relationship is of immense practical help to us in our mathematical work. And this is with regard to the use of successive differentials for the detection of repeated factors.

The procedure hereof is so simple that it needs no elaborate exposition at all. The following examples will serve to show the *modus operandi* in question:

(1) Factorise x^3-4x^2+5x-2

$\therefore \frac{dy}{dx}=3x^2-8x+5=(x-1)(3x-5)$

Judging from the first and the last coefficients of E(the given expression), we can rule out $(3x-5)$ and keep our eyes on $(x-1)$.

$\therefore D_2=6x-8=2(3x-4) \quad \therefore$ we have $(x-1)^2$

$\therefore$ (According to the *Ādyam Ādyena* Sūtra) $E=(x-1)^2(x-2)$

(2) Factorise $4x^3-12x^2-15x-4$.

$\therefore D_1=12x^2-24x-15=3(4x^2-8x-5)=3(2x-5)(2x+1)$

$\therefore D_2=24x-24=24(x-1)$ $\therefore$ As before, we have, $(2x+1)^2$

$\therefore E=(2x+1)^2(x-4)$

(3) Factorise $x^4--6x^8+13x^2-24x+36$

$\therefore D_1=4x^3-18x^2+26x-24=2(2x^3-9x^2+13x-12)$

$=2(x-3)(2x^2-3x+4)$

$\therefore D_2=12x^2-36x+26$ (which has no rational factors)

$\therefore E=(x-3)^2(x^2+4)$

(4) Factorise: $2x^4-23x^3+84x^2-80x-64$

$\therefore D_1=8x^3-69x^2+168x-80$

$\therefore D_2=24x^2-138x+168=6(4x^2-23x+28)=6(x-4)(4x-7)$

$\therefore D_3=48x-138=6(8x-23)$

$\therefore D_2=6(x-4)(4x-7)$

$\therefore D_1=(x-4)^2(8x-5)$

$\therefore E=(x-4)^3(2x+1)$

(5) Resolve $x^4-5x^3-9x^2+81x-108$ into factors,

$\therefore D_1=4x^3-15x^2-18x+81$

$\therefore D_2=12x^2-30x-18=6(2x^2-5x-3)=6(x-3)(2x+1)$

$\therefore D_3=24x-30=6(4x-5)$

$\therefore D_2=(x-3)(12x+6)$

$\therefore D_1=(x-3)^2(4x+9)$

$\therefore E=(x-3)^3(x+4)$

(6) Resolve $16x^4-24x^2+16x-3$ into factors.

$\therefore D_1=64x^3-48x+16=16(4x^3-3x+1)$

$\therefore D_2=192x^2-48=48(4x^2-1)=48(2x-1)(2x+1)$

$\therefore D_3=384x$

$\therefore D_2=(2x-1)(96x+48)$

$\therefore D_1=(2x-1)^2(x+1)$

$\therefore E=(2x-1)^3(2x+3)$

(7) Resolve $x^5-5x^4+10x^3-10x^2+5x-1$ into factors.

$\therefore D_1=5x^4-20x^3+30x^2-20x+5$

$=5(x^4-4x^3+6x^2-4x+1)$

$\therefore D_2=20x^3-60x^2+60x-20=20(x^3-3x^2+3x-1)$

$\therefore D_3=3x^2-6x+3=3(x^2-2x+1)$

$\therefore\ D_4=6x-6=6(x-1)$
$\therefore\ D_3=3(x-1)^2$
$\therefore\ D_2=4(x-1)^3$
$\therefore\ D_1=5(x-1)^4$
$\therefore\ E\ =(x-1)^5$

(8) Factorise $x^5-15x^3+10x^2+60x-72$

$\therefore\ D_1=5x^4-45x^2+20x+60=5(x^4-9x^2+4x+12)$
$\therefore\ D_2=20x^3-90x+20=10(2x^3-9x+2)$
$\therefore\ D_3=60x^2-90=30\ (2x^2-3)$
$\therefore\ D_4=120x$;
$\therefore\ D_2=20(x-2)^2\ (x+1)$
$\therefore\ D_1=5(x-2)^2\ (x+1)\ (x+3)$
$\therefore\ E\ =(x-2)^3\ (x+3)^2$

Many other such applications are obtainable from the Vedic Sūtras relating to चलन-कलन (*Calana-Kalana*—Differential Calculus). They are, however, to be dealt with, later on.

CHAPTER XXIII

PARTIAL FRACTIONS

Another subject of very great importance in various mathematical operations in general and in Integral Calculus in particular is "Partial Fractions" for which the current systems have a very cumbrous procedure but which the *'Parāvartya'* **Sūtra** tackles very quickly with its well-known MENTAL ONE-LINE answer process.

We shall first explain the current method; and, along-side of it, we shall demonstrate the *"Parāvartya"* **Sūtra** application thereto. Suppose we have to express $\frac{3x^2+12x+11}{(x+1)(x+2)(x+3)}$ in the shape of Partial Fractions.

The current method is as follows:

Let— $\frac{3x^2+12x+11}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$

$$\therefore \frac{3x^2+12x+11}{(x+1)(x+2)(x+3)} = \frac{A(x^2+5x+6)+B(x^2+4x+3)+C(x^2+3x+2)}{(x+1)(x+2)(x+3)}$$

$$\therefore x^2(A+B+C)+x(5A+4B+3C)+(6A+3B+2C)=(3x^2+12x+11)$$

$\therefore$ Equating the coefficients of like powers on both sides,

$$\left.\begin{array}{r} A+B+C=3 \\ 5A+4B+3C=12 \\ 6A+3B+2C=11 \end{array}\right\}$$

$\therefore$ Solving these three simultaneous equations involving three unknowns, we have, $A=1$; $B=1$; and $C=1$

$$\therefore E = \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3}$$

In the Vedic system, however, for getting the value of A,

(i) we equate its denominator to zero and thus get the *Parāvartya* value of A(i.e. -1);

(ii) and we MENTALLY substitute this value -1 in the E, (but without the factor which is A's denominator on the R.H.S.) &

(iii) we put this result down as the value of A. Similarly for B and C.

Thus, $A=\frac{3x^2+12x+11}{(x+2)(x+3)}=\frac{3-12+11}{1\times 2}=1$;

$B=\frac{3x^2+12x+11}{(x+1)(x+3)}=\frac{12-24+11}{(1)(-1)}=\frac{-1}{-1}=1$;

and $C=\frac{3x^2+12x+11}{(x+1)(x+2)}=\frac{27-36+11}{(-2)(-1)}=\frac{2}{2}=1$

$\therefore\ E=\frac{1}{x+1}+\frac{1}{x+2}+\frac{1}{x+3}$

Note:—All this work can be done *mentally*; and all the laborious work of deriving and solving three simultaneous equations is totally avoided by **this** method.

A few more illustrations are shown below:

(1) $\frac{2x+3}{(x+1)(x+2)}=\frac{1}{x+1}+\frac{1}{x+2}$ (also available by mere *Vilokanam*)

(2) $\frac{7}{(x+1)(x+2)}=\frac{7}{x+1}-\frac{7}{x+2}$

(3) $\frac{2x-5}{(x-2)(x-3)}=\frac{1}{x-2}+\frac{1}{x-3}$

(4) $\frac{3x+13}{(x+1)(x+2)}=\frac{10}{x+1}-\frac{7}{x+2}$

(5) $\frac{2x+1}{x^2-5x+6}=\frac{-5}{x-2}+\frac{7}{x-3}$

(6) $\frac{7x-1}{1-6x+6x^2}=\frac{-6}{1-2x}+\frac{4}{1-3x}$

(7) $\frac{9}{x^2+x-2}=\frac{3}{x-1}-\frac{3}{x+2}$

(8) $\frac{x-13}{x^2-2x-15}=\frac{2}{x+3}-\frac{1}{x-5}$

(9) $\frac{x-5}{x^2-x-2}=\frac{2}{x+1}-\frac{1}{x-2}$

(10) $\frac{x+37}{x^2+4x-21}=\frac{4}{(x-3)}-\frac{3}{x+7}$

(11) $\frac{5+2x-3x^2}{(x^2-1)(x+1)}=\frac{(1+x)(3-3x)}{(x+1)^2(x-1)}$

$=\frac{5-3x}{x^2-1}=\frac{1}{x-1}-\frac{4}{x+1}$

(12) $\frac{5x-18}{x^2-7x+12}=\frac{3}{x-3}+\frac{2}{x-4}$

(13) $\frac{3x^2-10x-4}{(x^2-6x+8)}=3+\frac{8x-28}{(x-2)\,(x-4)}=3+\frac{6}{x-2}+\frac{2}{x-4}$

(14) $\frac{x^2+x+9}{x^3+6x^2+11x+6}=\frac{9}{2(x+1)}-\frac{11}{x+2}+\frac{15}{2(x+3)}$

(15) $\frac{2x+1}{x^3-6x^2+11x-6}=\frac{3}{2(x-1)}-\frac{5}{x-2}+\frac{7}{2(x-3)}$

(16) $\frac{2x^3-11x^2+12x+1}{x^3-6x^2+11x-6}=2+\frac{x^2-10x+13}{(x-1)\,(x-2)\,(x-3)}$

$$=2+\frac{2}{x-1}+\frac{3}{x-2}-\frac{4}{x-3}$$

Therefore, the GENERAL FORMULA is :

$$\frac{6x^2+mx+n}{(x-a)\,(x-b)\,(x-c)}$$

$. A=\frac{la^2+ma+n}{(a-b)(a-c)}$; $B=\frac{lb^2+mb+n}{(b-c)(b--a)}$; and $C=\frac{lc^2+mc+n}{(c-a)(c-b)}$

If and when, however, we find one or more factors of the Denominator in repetition (i.e. a square, a cube etc,,) a slight variation of procedure is obviously indicated. For example, let E be $\frac{3x+5}{(1-2x)^2}$

According to the current system, we say :

Let $1-2x=p$ (sothat $x=\frac{1-p}{2}$)

$$\therefore E=\frac{\frac{3-3p}{2}+5}{p^2}=\frac{13-3p}{2p^2}$$

$$=\frac{13}{2p^2}-\frac{3}{2p}=\frac{13}{2(1-2x)^2}-\frac{3}{2(1-2x)}$$

This is no doubt a straight and simple procedure. Rut even this is rather cumbrous, certainly not easy and certainly not mental arithmetic ! And, with bigger numbers and higher numbers (as will be the case in the next example), it will he still worse !

The Vedic system, however, gives us two very easy *Parāvartya* methods whereby the whole work can be done mentally, easily and speedily. They are as follows:

(i) $\frac{3x+5}{(1-2x)^2}=\frac{A}{(1-2x)^2}+\frac{B}{1-2x}$

$\therefore$ $3x+5=A+B(1-2x)$..............M

$\therefore$ $-2Bx=3$: and $A+B=5$

$\therefore$ $B=-1\frac{1}{2}$ and $A=6\frac{1}{2}$

(ii) $3x+5=A+B(1-2x)$..............M

By *Parāvartya* (making $1-2x=0$, i.e. $x=\frac{1}{2}$),

we have $A=6\frac{1}{2}$; and (as this is an absolute identity) (i.e true for all values of x), let us put $x=0$

$\therefore$ $A+B=5$ $\therefore$ $B=-1\frac{1}{2}$

Two more examples are taken by way of Illustration :

(1) $\frac{x^3+3x+1}{(1-x)^4}$

According to the current system, we say:

let $1-x=p$ (so that $x=1-p$)

$$\therefore E=\frac{(1-p)^3+3(1-p)+1}{p^4}$$

$$=\frac{1-3p+3p^2-p^3+3-3p+1}{p^4}$$

$$=\frac{5}{p^4}-\frac{6}{p^3}+\frac{3}{p^2}-\frac{1}{p}$$

$$=\frac{5}{(1-x)^4}-\frac{6}{(1-x)^3}+\frac{3}{(1-x)^2}-\frac{1}{(1-x)}$$

But according to the Vedic procedure, we say :

(1) $A+B(1-x)+C(1-x)^2+D(1-x)^3=x^3+3x+1$

$\therefore$ $(A+B+C+D)+x(-B-2C-3D)+x^2(C-3D)-Dx^3$
$-x^2+3x+5$

$\therefore$ $-D=1$ $\therefore$ $D=-1$

$\therefore$ $C-3D=C-3=0$ $\therefore$ $C=3$

$\therefore$ $-B-2C-3D=-B-6+3=3$ $\therefore$ $B=-6$

$\therefore$ $A+B+C+D=A-6+3-1=1$ $\therefore$ $A=5$

Or, secondly, (by ***Parāvartya***),

Put $x=1$ $\qquad\therefore A=5$

Put $x=0$ $\therefore A+B+C+D=1$ $\qquad\therefore H=-6$

Put $x=2$ $\therefore A-B+C-D=15$ $\qquad C=3$

$\therefore D=-1$

(all of which can be done by *mental* Arithmetic).

(2) $$\frac{5+2x-2x^2}{(x^2-1)(x+1)}=\frac{A}{(x+1)^2}+\frac{B}{x+1}+\frac{C}{x+1}$$

$$\therefore A(x-1)+B(x^2-1)+C(x-1)^2$$
$$=(-A-B+C)+x(A+2C)+x^2(B+C)$$
$$=5+2x-2x^2$$

$\therefore A+BC-=-5$; $A+2C=2$; $B+C=-2$

$\therefore A=-\frac{1}{2}$; $B=-3\frac{1}{4}$; and $C=1$

or, secondly (by ***Parāvartya***),

Put $x=1$ $\therefore 4c=5$ $\quad\therefore C=1\frac{3}{4}$

$x=1$ $\therefore -2A=1$ $\quad\therefore A=-\frac{1}{2}$

$x=0$ $\quad\therefore B=-3\frac{1}{4}$

N.B. :—1. It need be hardly pointed out that the current method will involve an unquestionably cumbrous and clumsy process of working, with all the attendant waste of time, energy etc.

2. Other details of applications of ***Parāvartya*** and other **Sūtras** to partial fractions, will be -dealt with later.

3. Just now we take up an important part of Integral Calculus wherein, with the help of partial fractions, we can **easily** perform difficult integrational work.

CHAPTER XXIV

INTEGRATION

By

PARTIAL FRACTIONS

In this chapter we shall deal, briefly, with the question of INTEGRATION by means of Partial Fractions. But, before we take it up, it will not be out of place for us to give a skeleton-sort of summary of the first principles and process of integration (as dealt with by the *Ekādhika Sūtra*).

The original process of differentiation is, as is wellknown, a process in which we say :

Let $y=x^3$. Then D_1 (i.e. $\frac{dy}{dx}$) $=3x^2$;

$D_2=6x$; and $D_3=6$

Now, in the converse process, we have :

$\frac{dy}{dx}=3x^2 \therefore dy=3x^2dx$

Integrating, $\therefore \int dy=\int 3x^2dx \quad \therefore y=x^3$

Thus, in order to find the integral of a power of x, we add unity to the पूर्व (*Pūrva* i.e. the original index) and divide the coefficient by the new index (i.e. the original one plus unity).

A few specimen examples may be taken :

(1) Integrate $28x^3$. $\int 28x^3dx=\frac{28}{4}x^4=7x^4$

(2) $\int(x^4+3x^3+6x^2+7x-9)dx$
$=\frac{1}{5}x^5+\frac{3}{4}x^4+2x^3+3\frac{1}{2}x^2-9x+K$(where K is an independent term)

$3\int(x^a+x^{a-1}+x^{a-2}$ etc.)

(3) $=\frac{x^{a+1}}{a+1}+\frac{x^a}{a}+\frac{x^{a-1}}{a-1}$......Etc.

(4) $\int(ax^{m+1}+bx^{m}+cx^{m-1})dx$

$$=\frac{ax^{m+2}}{m+2}+\frac{bx^{m+1}}{m+1}+\frac{cx^{m}}{m}\text{......Etc.}$$

This is simple enough, so far as it goes. But what about complex expressions involving numerators and denominators? The following sample specimens will make the procedure (by means of Partial Fractions) clear:

(1) Integrate $\frac{7x-1}{6x^2-5x+1}$

$\because$ (By *Parāvartya*), $\frac{7x-1}{6x^2-5x+1}=\frac{7x-1}{(2x-1)(3x-1)}$

$$=\frac{5}{2x-1}-\frac{4}{3x-1}$$

$$\therefore \int\frac{(7x-1)dx}{6x^2-5x+1}=5\int\frac{dx}{2x-1}-4\int\frac{dx}{3x-1}$$

$$=\tfrac{5}{2}\int\frac{d(2x)}{2x-1}-\tfrac{4}{3}\int\frac{d(3x)}{3x-1}$$

$$=\tfrac{5}{2}\log(2x-1)-\tfrac{4}{3}\log(3x-1)$$

$$=\log\frac{(2x-1)^{5/2}}{(3x-1)^{4/3}}$$

(2) Integrate $\frac{x^2-7x+1}{x^3-6x^2+11x-6}$

$\because$ (By *Parāvartya*), $\frac{x^2-7x+1}{x^3-6x^2+11x-6}=\frac{x^2-7x+1}{(x-1)(x-2)(x-3)}$

$$=\frac{-5}{2(x-1)}+\frac{9}{x-2}-\frac{11}{2(x-3)}$$

$$\therefore \int\frac{(x^2-7x+1)dx}{x^3-6x^2+11x-6}=\int\left\{\frac{-5}{2(x-1)}+\frac{9}{x-2}-\frac{11}{2(x-3)}\right\}dx$$

$$=-\tfrac{5}{2}\int\frac{dx}{x-1}+9\int\frac{dx}{x-2}-\frac{11}{2}\int\frac{dx}{x-3}$$

$$=-\tfrac{5}{2}\log(x-1)+9\log(x-2)-\tfrac{11}{2}\log(x-3)$$

(3) Integrate $\frac{1}{x^3-x^2-x+1}$

Let $\frac{1}{x^3-x^2-x+1}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}$......M

$\therefore$ $1=A(x-1)(x+1)+B(x+1)+C(x-1)^2$

$=A(x^2-1)+B(x+1)+C(x-1)^2$..............N

Now, let $x=1$ $\therefore$ $1=2B$ $\therefore$ $B=\frac{1}{2}$

Differentiating $(\overline{\underline{N}})$,

$0=2Ax+B+2Cx-2c$..............P

Now put $x=1$ $\therefore$ $2A=-\frac{1}{2}$ $\therefore$ $A=-\frac{1}{4}$

Differentiating $(\overline{\underline{P}})$, $2A+2C=0$ $\therefore$ $2C=\frac{1}{2}$ $\therefore$ $C=\frac{1}{4}$

$$\therefore E=\frac{-1}{4(x-1)}-\frac{1}{2(x-1)}+\frac{1}{4(x+1)}$$

$$\therefore \int\frac{dx}{x^3-x^2-x+1}=-\frac{1}{4}\int\frac{dx}{(x-1)}=-\frac{1}{2}\int\frac{dx}{(x-1)^2}+\frac{1}{4}\int\frac{dx}{(x+1)}$$

$$=-\frac{1}{4}\log(x-1)=\frac{1}{2}\log(x-1)+\frac{1}{4}\log(x+1)$$

CHAPTER XXV

THE VEDIC NUMERICAL CODE

It is a matter of historical interest to note that, in their mathematical writings, the ancient Sanskrit writers do not use figures (when big numbers are concerned) in their numerical notations but prefer to use the letters of the Sanskrit (Devanagari) alphabet to represent the various numbers! And this they do, not in order to conceal knowledge but in order to facilitate the recording of their arguments, and the derivation conolusions etc. The more so, because, in order, to help the pupil to memorise the material studied and assimilated they made it a general rule of practice to write even the most technical and abstruse text-books in Sūtras or in Verse (which is so much easier—even for the children—to memorise) than in prose (which is so much harder to get by heart and remember). And this is why we find not only theological, philosophical, medical, astronomical and other such treatises but even huge big *dictionaries* in Sanskrit Verse! So, from this stand-point, they used verse, Sūtras and codes for lightening the burden and facilitating the work (by versifying scientific and even mathematical material in a readily assimilable form)!

The very fact that the alphabetical code (as used by them for t'his purpose) is in the natural order and can be immediately interpreted, is clear proof that the code language was resorted not for concealment but for greater ease in verification etc., and the key has also been given in its simplest form:

"कादि नव, टादि नव, पादि पञ्चक, याद्यष्टक and a: शून्यम्" which means:

(1) *ka* and the following *eight* letters;
(2) *ta* and the following *eight* letters;
(3) *pa* and the following *four* letters;
(4) *ya* and the following *seven* letters; and
(5) *kṣa* (or Kṣudra) for Zero.

Elaborated, this means:

(1) *ka, ta, pa* and *ya* all denote 1;
(2) *kha, tha, pha* and *ra* all represent 2;

(3) *ga*, *ḍa*, *ba* and *la* all stand for **3**;
(4) *gha*, *dha*, *bha* and *va* all denote **4**;
(5) *gna*, *ṇa*, ma and *sa* all represent **5**;
(6) *ca*, *ta*, and *śa* all stand for 6;
(7) *cha*, *tha*, and *ṣa* all denote 7;
(8) *ja*, *da* and *ha* all represent 8;
(9) *jha* and *dha* stand for 9; and
(10) *Kṣa* (or *Kṣudra*) means Zero!

The vowels (not being included in the list) make no difference; and in conjunct consonants, the last consonant is alone to be counted.

Thus *pa pa* is 11, *ma ma* is 55, *ṭa ṭa* is 11, *ma ry* is 52 and so on!

And it was left to the author to select the particular consonant or vowel which he would prefer at each step. And, generally, the poet availed himself of this latitude to so frame his selections as to bring about another additional meaning (or meanings) of his own choice. Thus, for instance, *kapa*, *ṭapa*, *papa* and *yapa* all mean 11; and the writer can (by a proper selection of consonants and vowels) import another meaning also into the same verse. Thus "I want *mama* and *papa*" will mean "*I* want 55 and 11"!

Concrete, interesting and edifying illustrations will be given later on (especially in connection with recurring decimals, Trigonometry etc. wherein, over and above the mathematical matter on hand, we find historical allusions, political reflections, devotional hymns in praise of the Lord Shri Krishna, the Lord Shri Shankara and so on!)*

This device is thus not merely a potent aid to versification (for facilitating memorisation) but has also a humorous side to it (which adds to the fun of it)!

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*The hymn (in praise of the Lord) gives us the value of $\frac{\pi}{10}$ to 32 decimal places (in Trigonometry).

CHAPTER XXVI

RECURRING DECIMALS

It has become a sort of fashionable sign of cultural advancement, not to say up-to-datism, for people now-a-days to talk not only grandly but also grandiosely and grandiloquently about Decimal coinage, Decimal weights, Decimal measurements etc. ; but there can be no denying or disguising of the fact that the western world as such—not excluding its mathematicians, physicists and other expert scientists-seems to have a tendency to *theorise* on the one hand on the superiority of the decimal notation and to fight shy, on the other, in actual *practice*—of decimals and positively prefer the "vulgar fractions" to them!

In fact, this deplorable state of affairs has reached such a pass that the mathematics syllabus—curricula in the schools, colleges and universities have been persistently "progressing" and "advancing" in this wrong direction to the extent of declaring that Recurring decimals are not integral parts of the matriculation course in mathematics and actually instructing the pupils to convert all recurring decimals AT SIGHT into their equivalent vulgar fraction shape, complete the whole work with them and finally re-convert the fraction result back into its decimal shape!

Having invented the zero mark and the decimal notation and given them to the world (as described already from the pages of Prof. Halstead and other Historians of Mathematics), the Indian Vedic system has, however, been advocating the decimal system, not on any *apriori* grounds or because of partiality but solely on its intrinsic *merits.* Its unique achievements in this direction have been of a most thrillingly wonderful character : and we have already—at the very commencement of

this illustrative volume—given a few startling **sample-specimens** thereof. The **student** will doubtless remember that, at the end of that chapter, we promised to go **into** fuller details of this subject at a **later** stage. In fulfilment of that promise, we now pass on to a further exposition of the marvels **of** Vedic mathematics in this direction.

Preliminary Note

We may begin this part of this work with a brief reference to the well-know1 distinction **between** non-recurring decimals, recurring ones and partly-recurring ones.

(i) A denominator containing only **2** or **5** as factors gives us an ordinary (i.e. **non-recurring** or **non-circulating)** decimal fraction (each 2, 5 or **10** contributing ***one*** significant digit to the decimal). For instance,

$$\frac{1}{2}=\cdot 5\ ;\ \frac{1}{4}=\frac{1}{2\times 2}=\cdot 25\ ;\ \frac{1}{8}=\frac{1}{2\times 2\times 2}=\cdot 125\ ;$$

$$\frac{1}{16}=\frac{1}{2\times 2\times 2\times 2}=\cdot 0625\ ;\ \frac{1}{32}=\frac{1}{2^5}=\cdot 03125\ ;$$

$$\frac{1}{5}=\cdot 2\ ;\ \frac{1}{10}=\cdot 1\ ;\ \frac{1}{20}=\cdot 05\ ;\ \frac{1}{25}=\frac{1}{5^2}=\cdot 04\ ;$$

$$\frac{1}{40}=\frac{1}{10\times 2^2}=\cdot 025\ ;\ \frac{1}{50}=\frac{1}{10\times 5}=\cdot 02\ ;$$

$$\frac{1}{80}=\frac{1}{10\times 2^3}=\cdot 0125\ ;\ \frac{1}{100}=\frac{1}{10^2}=\cdot 01\ ;\ \text{and so on.}$$

(ii) Denominators containing only **3,** 7, **11** or higher prime numbers as factors (and not even a single 2 or 5) give **us** recurring (or circulating) decimals whicn we shall deal with in detail in this chapter and in some other later chapters too.

$$\frac{1}{3}=\cdot\dot{3}\ ;\ \frac{1}{7}=\cdot\dot{1}4285\dot{7}\ ;\ \frac{1}{9}=\cdot\dot{1}\ ;\ \frac{1}{11}=\cdot\dot{0}\dot{9}\ ;$$

$$\frac{1}{13}=\cdot\dot{0}7692\dot{3}\ ;\ \frac{1}{17}=\cdot\dot{0}5882352/941176\dot{4}7\ ;$$

$$\frac{1}{19}=\ 052631578/947368421\ ;\ \text{and so on.}$$

(iii) A denominator with factors partly of the former type (i.e. 2 and 5) and partly of the latter type **(i.e. 3, 7, 9 etc.,)** gives **us** a mixed **(i.e.** partly recurring and partly non-recurring)

decimal, (each 2, 5 or 10 contributing one non-recurring digit to the decimal).

$\frac{1}{6}=\frac{1}{2\times3}=\cdot1\dot{6}$; $\frac{1}{15}=\frac{1}{3\times5}=\cdot0\dot{6}$;

$\frac{1}{18}=\frac{1}{2\times9}=\cdot0\dot{5}$; $\frac{1}{22}=\frac{1}{2\times11}=\cdot0\dot{4}\dot{5}$;

$\frac{1}{24}=\frac{1}{2^3\times3}=\cdot041\dot{6}$; and so on.

N.B. :—(i) Each 3 or 9 contributes only one recurring digit ; 11 gives 2 of them ; 7 gives 6 ; and other numbers make their own individual contribution (details of which will be explained later).

(ii) In every non-recurring decimal with the standard numerator (i.e. 1), it will be observed that the last digit of the denominator and the last digit of the equivalent decimal, multiplied together, will always yield a product ending in zero ; and

(iii) In every recurring decimal x with the standard numerator (i.e. 1), it will be similarly observed that 9 will invariably be the last digit of the product of the last digit of the denominator and the last digit of its Recurring Decimal Equivalent (nay, the product is actually a continuous series of NINES) !

Thus, $\frac{1}{2}=\cdot5$; $\frac{1}{5}=\cdot2$; $\frac{1}{10}=\cdot1$; $\frac{1}{4}=\cdot25$; $\frac{1}{8}=\cdot125$;
$\frac{1}{16}=\cdot0625$; $\frac{1}{25}=\cdot04$; $\frac{1}{125}=\cdot008$; etc.

And $\frac{1}{3}=\cdot\dot{3}$; $\frac{1}{7}=\cdot\dot{1}4285\dot{7}$; $\frac{1}{9}=\cdot\dot{1}$; $\frac{1}{11}=\cdot\dot{0}\dot{9}$;
$\frac{1}{13}=\cdot\dot{6}7692\dot{3}$; etc., etc.

And this enables us to determine beforehand, the last digit of the recurring decimal equivalent of a given vulgar fraction. Thus $\frac{1}{17}$ in its decimal shape must necessarily epd in 7 ; $\frac{1}{19}$ in 1 ; $\frac{1}{21}$ in 9 ; $\frac{1}{23}$ in 3 ; and so on. The immense practical utility of this rule in the conversion of vulgar fractions into their decimal shape has already been indicated in the first chapter and will be expatiated on, further ahead in this chapter and in subsequent chapters.

Let us first take the case of $\frac{1}{7}$ and its conversion from the vulgar fraction to the decimal shape. We note here:

$7)1.0(.\dot{1}4285\dot{7}$
7
30
28
20
14
60
56
40
36
60
49
1

(i) that the successive remainders are 3, 2, 6, 4, 5 and 1 and that, inasmuch as 1 is the original figure with which we started, the same remainders are bound to repeat themselves in the same sequence (endlessly). And this is where we stop the division-process and put the usual recurring marks (the dots on the first and the last digits) in order to show that the decimal has begun its characteristic (recurring) character.

At this point, we may note that inasmuch as the first dividend 10 (when divided by 7) gives us the first remainder 3, and, with a zero affixed to it, this 3 will (as 30) become our second dividend and inasmuch as this process will be continuing indefinitely (until a remainder repeats itself and warns us that the recurring decimal's recurring character has begun to manifest itself, it stands to reason that there should be a uniform ratio in actual action. In other words, because the first Dividend 10 gives us the first remainder 3 and the second dividend 30, therefore (*Āanurūpyeṇa* i.e. according to the ratio in question or by simple rule of three), this second dividend 3 should give us the second remainder 9! In fact, it is a "*Geometrical* Progression" that we are dealing with!

And when we begin testing the successive remainders from this standpoint, we note that the said inference (about the Geometrical Progression with the common ratio 1 : 3) is correct For, although, when we look for 3 X 3=9 as the second remainder, we actually find 2 there instead, yet as 9 is greater than 7 (the divisor), it is but proper that, by further division of 9 by 7, we get 2 as the remainder. And then we observe that this second remainder (2) yields us the third remainder 6, and thereby keeps up the Geometrical Progression (with the same

ratio 1 : 3). In the same way, this *6* gives us *18* which (being greater than the divisor and being divided by it) gives us 4 as the fourth remainder. And 4 gives us *12* which (after division by 7) gives us 5 as the fifth remainder ! And, by the same ratio, this 5 gives us *15* which (when divided by 7) gives us 1 as the sixth Remainder. And as this was the dividend which we began with, we stop the division-process here !

The fun of the Geometrical Progression is no doubt there ; but it is not for the mere fun of it, but also for the practical utility of it, that we have called the student's attention to it. For, in the actual result, it means that, once we know the ratio between the first dividend and the first remainder (1 : 3 in the present case), we can—without actual further rlivision—automatically put down all the remainders (by maintaining the 1 : 3 Geometrical Progression). For example, in the present case, since the ratio is uniformly *1 : 3*, therefore the second remainder is 9 (which after deducting the divisor), we set down as 2 ; and so on (until we reach 1). Thus our chart reads as follows :

7)1.0(G.P. *1, 3, 2,* 6, 4, *5*
7

3

1, 3, 2, 6, 4, 5.

Yes ; but what do we gain by knowing the remainders beforehand (without actual division)? The answer is that, as soon as we get the first remainder, our whole work is practically over. For, since each remainder (with a zero affixed) automatically becomes the next dividend, we can mentally do this affixing at *sight*, mentally work out the division at each step and put down the quotient automatically (without worrying about the remainder) ! For, the remainder is already there in front of us !

Thus the remainders 1, *3,* 2, *6,* 4 and 5 give us the successive dividends *10, 30, 20, 60, 40* and *50* ; and, dividing these mentally. by *7,* we can go forward or backward and obtain all the quotient-digits *1,* 4, *2, 8,* 5, and *7.* And, as it is a pure circulating decimal, our answer is *112857* !

(Dividend digits)
1, 3, 2, 6, 4, 5
---
1 4 2 8 5 7
(Quotient-digits)

There is, however, a still more wonderful Vedic method by which, without doing even this little division-work, we can put down the quotient-digits automatically (forward or backward, from any point whatsoever)! The relevant Sūtra hereon says: शेषाण्यङ्केन चरमेण (*Śeṣāṇyaṅkena Carameṇa* and means: The Remainders by the last digit):

As explained in another context (in the very first chapter of this volume), the word by indicates that the operation is not one of addition or of subtraction but of division and of multiplication!

The division–process(whereby we affix a zero to 1, 3, 2 etc., divide the product by 7 and set down the quotient) has been shown just above. We now show the reverse process of multiplication, which is easier still.

In so doing, we put down not the dividend—nucleus digits but the remainders themselves in order: **3, 2, 6, 4, 5, 1.**

And, as we know from a previous paragraph that 7 is the last digit, we multiply the above-given remainders by 7 and put the last (i.e. the right-hand-most digit) down under each of the remainders (totally ignoring the other digit or digits, if any, of the product)! And lo! the answer is there in front of us again, (really looking more like magic than like mathematics)!

Thus,

(Remainders)
3, 2, 6, 4, 5, 1
1 4 2 8 5 7
(Quotient-digits)

3 multiplied by 7 gives us 21;
and we put down only 1;
2 × 7 gives us 14; and we put down only 4;
6 × 7 gives us 42; and we put down only 2;
4 × 7 gives us 28; and we put down only 8;
5 × 7 gives us 35; and we put down only 5; and
1 × 7 gives us 7; and we put down 7.

And the answer is $\cdot\dot{1}4285\dot{7}$

At **this** point, we may remind the student of a very important point which we have already explained in chapter I (regarding the conversion of $\frac{1}{17}$, $\frac{1}{19}$ and $\frac{1}{29}$ to their recurring decimal shape). This is in connection with the **facts** that the two halves of these decimals together total a series of **NINES**; that, once half the answer is known. the other half can be had by putting down the complements (from nine) of the digits already obtained; and that, as the ending of the first half of the result **synchronises** with our reaching of the difference between the numerator and the denominator as the remainder, we know when exactly we should stop the **division** (or **multi**plication, as the case may be) and begin the mechanical subtraction from 9 of the digits already found!

```
7)1·0 (·142
  7   ( 857
  ---
  30
  28
  ---
   20
   14
   ---
   60
   56
   ---
    40
    35
    ---
     60
     49
     ---
      1
```

The student **can** easily **realise** how, inasmuch as this rule is applicable to every **case** (wherein D ∽ N comes up as a **Remainder**), it therefore means an automatic reduction of even the little labour involved, by exactly one-half!

Going back to the original topic (re: the conversion of vulgar fractions into their equivalent decimal shape and how the Geometrical Progressional ratio can give us **beforehand**—without actual division—all the remainders that will come up in actual division), we now take up $\frac{1}{13}$ as another illustrative example and observe how the process works **out** therein :

(1) (i) $\frac{1}{13}$ Here the successive Dividends—**hucleus**—digits are 1, **10, 9, 12, 3** and 4. Affixing a zero to each of them and dividing the dividends by 13, we get **0**, 7, 6, 9, 2 and 3 as the first digits **of** the quotient in the answer.

```
13)1·00 (·076
    91  ( 923
    ---
     90   999
     78
     ---
     120
     117
     ---
      30
      26
       40
       39
```

(ii) Or, **secondly,** re-arranging the remainders so **as** to start from the first actual remainder, we have:

$$13)1\cdot00(\text{G.P. } 1, 10, 9, 12, 3, 4$$
$$0\ 7\ 6\ \ 9\ 2\ 3$$

10, 9, 12, 3, 4 and **1.** And multiplying these by **3(the last digit of the** answer in the present case), we put down merely the right—hand—most digit of each product; and these are the successive quotient—digits! Here too, as usual, we go forward or backward (or in any sequence which we may choose). And the answer is **$\cdot$076923**

$$13)1.00(10, 9, 12, 3, 4, 1$$
$$0\ 7\ \ 6\ 9\ 2\ 3$$

(iii) And here too we observe, in operation, the rule about complements from NINE! And it commences from the point st whioh we obtain 12 (the difference between the Numerator and the denominator) as the Remainder.

$$\begin{array}{r} 076 \\ \mathbf{923} \\ \hline 999 \end{array}$$

(iv) In the above charts, we may avoid big numbers **by using** a **minus** where a big number is threatened. Thus, instead of taking **3** (as in the case of **7**), we may take -3 as the common geometrical ratio and will find the Geometrical Progression intact; and naturally the product of each remainder-digit by the last digit remains intact too and gives us the same answer: **$\cdot\dot{0}7692\dot{3}$!**

$$\text{G.P. } 1, \frac{3}{10}, 9, -\frac{27}{12} \text{ etc.}$$

$$-\frac{3}{10}, 9, -\frac{27}{12}, -\frac{36}{3}, -\frac{9}{4}, -\frac{12}{1}$$
$$0\ \ 7\ \ \ 6\ \ \ 9\ \ 2\ \ 3$$

We pass now on to still another and easier method which comes under the *Ekādhikā Sūtra* which we have expounded and explained at sufficient length already (in the first chapter) and which therefore we need only summarise **and** supplement here but need not elaborate again.

The *Ekādhika Sūtra* (which **means** 'by the preceding one increased by **one')** has already been shown at work in a number of ways and in a number of directions and on a number of occasions and will similarly come into operation still further, in **many** more ways and in many more contexts.

Numbers ending in Nine

(i) If and when the last digit of the denominator is 9, we know beforehand that the equivalent recurring decimal ends in **1**.

(ii) In the case of $\frac{1}{19}$, the last but one digit is 1; we increase it by 1 and make it 2. In $\frac{1}{29}$ we work with 2+1=3. In $\frac{1}{39}$ and in $\frac{1}{49}$, we operate with 4 and 5 respectively and so on.

(iii) In the multiplication-process (by *Ekādhika Pūrva*), in all these cases, we put 1 down as the last digit (i.e. in the right-hand-most place); and we go on multiplying that last digit (1) from the right towards the left by 2, 3, 4 and 5 respectively; and when there is more than one digit in that product, we set the last of those digits down there and carry the rest of it over to the next immediately preceding digit towards the left.

(iv) When we get D ∽ N as the product, we know we have done half the work; we stop the multiplication there; and we mechanically put down the remaining half of the answer (by merely taking down the complements from NINE).

(v) The division-process (by *Ekādhika Sūtra*) follows the same rules (vide Supra).

(1) We may first consider the fraction $\frac{1}{19}$ as our first illustration of the method described:

(i) Putting 1 as the last digit and continually multiplying by 2 towards the left, we get the last four digits (towards the left) without the least difficulty.

. . . . 8 4 2 1
9 4 7 3 6

(ii) 8×2=16. Therefore put 6 down immediately to the left of 8 (with 1 to carry over). 6×2+the 1 carried over—13. Put the 3 to the left of the 6 (with 1 to carry over). 3×2+1=7 Set it down before the 3 (with nothing to carry over). 7×2=14. Therefore put the 4 before the 7 (with 1 to carry over) 4×2+the 1 carried over=9.

(iii) We have thus got 9 digits by continual multiplication from the right towards the left. And now 9×2=18 (which is D ∽ N). This means that half the work is over and that the earlier 9 digits are obtainable by putting down the complements

(from NINE) of the digits already determined. So, we have
$\frac{1}{19} = 0\cdot\dot{0}\,5\,2\,6\,3\,1\,5\,7\,8/9\,4\,7\,3\,6\,8\,4\,2\,\dot{1}$

(2) Let us now examine the case of $\frac{1}{29}$:

Begin with 1 (as usual) at the extreme right end and go on multiplying by 3 each time, "carrying over" the surplus digit or digits (if any) to the left (i.e. to be added to the next product to be determined. Thus, when we have obtained 14 digits i.e.

..................9 6 5 5 1 7 2 4 1 3 7 9 3 1 ,

we find that we have reached 28; we know we have done half the work; and we get the first 14 digits by simply subtracting each of the above digits from NINE.

·0 3 4 4 8 2 7 5 8 6 2 0 6 8 1

$\therefore \frac{1}{29} = 0\cdot\dot{0}\,3\,4\,4\,8\,2\,7\,5\,8\,6\,2\,0\,6\,8/96\,5\,5\,1\,7\,2\,4\,1\,3\,7\,9\,3\,\dot{1}$

(3) Next let us take $\frac{1}{39}$

Take 1 again at the extreme right end and continually multiply by 4 from the right to the left. Thus, we have:

$\frac{1}{39} = \cdot\dot{0}2564\dot{1}.$

Note in this case that ∵ 39 is a multiple of 3 and 13 and not a prime number (like 19 and 29) and ∵ 3 and 13 give only 1 and 6 recurring decimals, there is a difference in its behaviour i.e. that the two halves are not complementary with regard to 9 hut only in relation to 6! In fact, D ∽ N (i.e. 38) does not come up at all as an interim product (as 18 and 28 did). And so, the question of complements from 9 does not arise at all; and the decimal equivalent has only 6 figures (and not 38)!

The reason for this is very simple. As $\frac{1}{39} = \frac{1}{13} \div 3$, as $\frac{1}{13}$ has only 6 recurring decimals in its decimal equivalent and because, for reasons to be explained a little later, this decimal equivalent of $\frac{1}{13}$ is exactly divisible by 9, much more therefore is it divisible by 3. And, consequently, when we divide it by 3 and exhaust the six digits, we find that there is no remainder left. In other words, $\frac{1}{39}$ has only 6 digits in its recurring decimal shape.

These have been obtained by the self-same *Ekādhika* process as served our purpose in the case of & and $\frac{1}{29}$

We next take up **and examine** the case of $\frac{1}{49}$ which, besides **following** the rules hereinabove explained, has the additional merit of giving us the clue to a still easier process for the conversion of vulgar fractions into their recurring decimal shape:

(1) $\frac{1}{49}=\frac{1}{7\times7}=\frac{\cdot\dot{1}4285\dot{7}}{7}$

(i) If ∴ we go on dividing 1 by **49** or $.\dot{1}4285\dot{7}$ by **7** (until the decimal begins to recur), we shall doubtless get our answer. But this will mean 42 steps of laborious working and is therefore undesirable.

(ii) We therefore adopt either of the *Ekādhika* methods and go on multiplying from right to left by 5 or dividing from left to right by 5.

(iii) On completing 21 digits, we find 48 (i.e. D ∽ N) coming up and standing up before us; and we mechanically put down the other **21** digits as usual (by the subtraction, from 9, of the digits already obtained). And the answer is:

$$\frac{1}{49}=\left.\begin{matrix}0\cdot0\,2\,0\,4\,0\,8\,1\,6\,3\,2\,6\,5\,3\,0\,6\,1\,2\,2\,4\,4\,8\\ 9\,7\,9\,5\,9\,1\,8\,3\,6\,7\,3\,4\,6\,9\,3\,8\,7\,7\,5\,5\,\dot{1}\end{matrix}\right\}$$

(iv) And this gives us the clue just above referred to about a still easier method (than even the *Ekādhika* ones) for the conversion of vulgar fractions **into** recurring decimals. And it **is** as follows:

By actual division (of 1 by **49**), we observe that the successive remainders are in Geometrical Progression (with the common ratio 1 : 2) that the dividends are similarly related and that each set of two digits in the quotient is also so related to its **predecessor**. **In other** words, **this** connotes and implies that, after putting down 02, we can **automatically** put down 04, 08, **16** and **32** and so on.

```
49)1·00( ·20408
     98
    ----
     200
     196
     ---
      400
```

But when we reach 64, we **find** that **$2 \times 64 = 128$ i.e. has** 3 **digits.** All that we have to do then is to add the 1 of the 128 over to the 64 already there, turn it into 65 and then put down not **28** but the remaining part of double the ***corrected*** **figure 65** (**i.e.** 30) and carry the process **carefully** on to the very end (**i.e.** until the decimal starts to recur). We therefore have :

$$\frac{1}{49} = \left.\begin{array}{l} 0\sim 0\;2\;0\;4\;0\;8\;1\;6\;3\;2\;6\;5\;3\;0\;\underline{6}\;1\;2\;2\;4\;4\;8 \\ \underline{9}\;7\;\underline{9}\;5\;\underline{9}\;1\;8\;3\;6\;7\;\underline{3}\;4\;6\;9\;\underline{3}\;8\;\underline{7}\;7\;\underline{5}\;5\;\underline{\dot{1}} \end{array}\right|$$

This new method does not apply to all cases but only to some special cases where the Denominator of the given vulgar fraction (or an integral multiple thereof) is very near a power of ten and thus lends itself to this kind of treatment. In such cases, however, it is the best procedure of all.

Note :—The rule of complements (from 9) is actually at work in this case too ; but, inasmuch as (for reasons to be explained hereafter), the **actual** total number of digits is 42, the first half of **it** ends with the **21st** digit and as we have been taking **up** a **group of** two digits at each step, we naturally by-pass the **21st** digit (which is concealed, so to speak, in the middle of the **11th** group). But, even then, the double-digit process is **so very** simple that continuation thereof can present no difficulty.

Other Endings

So far, we have considered only vulgar fractions whose denominators end in 9. Let us now go on to and study the cases of $\frac{1}{7}$, $\frac{1}{13}$, $\frac{1}{17}$, $\frac{1}{23}$ and other such fractions (whose denominators end not in 9 but in 1, 3 or 7).

(i) **Here** too, we first make up our minds, at sight, as regards the last **digit** of the decimal equivalent. Thus, Denominators ending in 7, **3** and **1** must necessarily yield decimals ending in 7, **3** and 9 (so that the product of the last digit of the denominator and the last digit of the decimal equivalent may end in 9.) Let **us** start with the case of $\frac{1}{7}$.

(ii) Put down $\frac{1}{7}$ in the shape $\frac{7}{49}$ $\qquad \frac{1}{7}=\frac{7}{49}$

(iii) Take 5 (one more than 4) as the ***Ekādhika Pūrva*** for the required multiplication or division (as the case may be).

(iv) Thus start with 7 at the right end

8 5 7
2 3

(v) Multiply it by 5 and set down 35 as shown in the marginal chart.

(vi) Multiply 5 by 5, add the 3 to the product and set 28 down in the same way.

Now, $5 \times 8+2=42$. But that is D ∽ N. Therefore put 142 down as the first half (accordingto the complements rule) $\therefore \frac{1}{7}=\cdot 1\dot{4}2/85\dot{7}$

142 | 8 5 7
| 2 5

Or $\frac{1}{7}=\frac{7}{49}$

The ***Ekādhika*** being 5, divide 7 by 5 and continue the division as usual (with the same rule of procedure).

i 4 2 / 8 5
2 1 4 /

After getting the three quotient-digits 1, 4 and 2 you find 42 as the remainder before you. So tackle the last 3 digits (according to the complements rule) and say:

$\frac{1}{7}=\frac{7}{49}=\cdot 142/857$

(2) Let us now take the case of $\frac{1}{13}=\frac{3}{39}$:

(i) The last digit is 3∴the last digit (in the answer) will be 3.

·0 7 6 / 9 2 3
·0 7 6 / 9 2 3
2 3

(ii) The ***Ekādhika*** (multiplier or divisor) is ∴4.

(iii) $\frac{1}{13}=\frac{3}{39}$. After 3 digits (whether by multipliction or by division), 36 (D ∽ N) comes up. So, the other half is mechanically set down. And we say:

$\frac{1}{13}=\cdot\dot{0}76/92\dot{3}$

(3) Next, let us take $\frac{1}{11}=\frac{9}{99}$

(i) The last digit is **1.** The last digit (of the answer) will be 9.

$\dot{0}$ / $\dot{9}$
9 /

(ii) The ***Ekādhika*** (in both ways) is **10.**

(iii) Immediately after the very first digit, we get **90** (which is **90** ∽ 9) before us. So, the complementst rule operates. . .

(iv) And, in either case, we get $\frac{1}{11}=\cdot 0/9$

(4) $\frac{1}{23}=\frac{3}{69}$ (giving **7** as ***Ekādhika***) and **3** as the last digit of the answer.

∴ (By both methods, Multiplication and Division).

Is = $0\cdot\dot{0}4347826086$ / $9565217391\dot{3}$

(5) $\frac{1}{17}=\frac{7}{119}$ (giving 12 as Ekidhika and **7** as the last digit)

∴ By both the methods (multiplication and division). we have:

$\frac{1}{17}=\frac{7}{119}=\cdot\dot{0}5882352/941176\dot{4}7$

The Code Language at *Work.*

Not only do the Vedic Sūtras tell us how to do all this by easy and rapid processes of mental arithmetic; but they have also tabulated the results in the shape of special sub-Sūtras (containing merely illustrative specimens with a master-key for "unlocking other portals" too). The abstruser details (and the master-key) are not given here ; but a few sample-specimens are given of the way in which the code and the ***Ekanyūna Sūtra*** (explained in Chapter 2) can be utilised for the purpose of postulating mental one-line answers to the questions in question. The three samples read as follows:

(1) केवलैः सप्तकं गुण्यात् (*Kevalaiḥ Saptakaṃ Guṇyāt*) ;

(2) कलौ क्षुद्रससैः (*Kalau Kṣudrasasaiḥ*) : and

(3) कंसे क्षामदाहखलैर्मलैः (*Kaṃse Kṣāmadāha-khalairmalaiḥ*)

In the first of these, ***Saptaka*** means 'seven' ; and ***Kevalaiḥ*** represents **143** ; and we are told that, in the case of seven, our multiplicand should be **143** !

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In the second, *Kalau* means 13 and ***Kṣudrasasaiḥ*** represents 077 ; and we are told that the multiplicand should be 077 ! and,

In the third, ***Kaṃse*** means 17 ; and ***Kṣāma-dāha-khalairmalaiḥ*** means 05882353 ; and we are told that the multiplicand should be this number of 8 digits !

Now, if we advert to the "*Ekanyūna*" corollary of the ***Nikhilam*** chapter (on multiplication), we shall be able to remind oursleves of the operation in question and the result to be achieved thereby. Let us do the multiplications accordingly (as directed) and see what happens.

(1) In the case of 7 (as denominator), $143\times999=142/857$; and lo ! these are the six recurring decimal digits in the answer !

(2) In the case of 13,$077\times999=076/923$; and these are the six digits in the recurring decimal equivalent of $\frac{1}{13}$! and

(3) In the case of 17, $05882353\times99999999=05882352/94117647$; and these are the 16 'recurring digits in the recurring decimal equivalent of $\frac{1}{17}$!

In all the 3 cases we observe the Rule of Complements (from 9) at work. And the sub-Sūtra merely gives us the necessary clue to the first half of the decimal and also a simple device (***Ekanyūnena***) for arriving at the whole answer ! And all this is achieved with the help of the easy alphabet-code !

These results may therefore be formulated as follows :

$$\frac{1}{7}=\frac{143\times999}{999999}=\frac{142857}{999999}=\cdot\dot{1}4285\dot{7}\ ;$$

$$\frac{1}{13}=\frac{077\times999}{999999}=\frac{076923}{999999}=\cdot\dot{0}7692\dot{3}\ ;\ \text{and}$$

$$\frac{1}{17}=\frac{05882353\times99999999}{9999999999999999}$$

$$=\cdot\dot{0}5882352/9411764\dot{7}\ !$$

And, by CROSS-multiplication, we get from the above, the following results :

(1) $7\times142857=999999$;

(2) $13\times076923=999999$; and

(3) 17×0588235294117647
$=9999999999999999$ (16 digits in all) !

And, just in passing, we may note that this is the reason why, in the case of all these vulgar fractions, the last digit of the denominator (9, 3, 7 or 1, as the case may be) gives, 1,3,7 or 9 before-hand as the last digit of the equivalent recurring decimal fraction!

The binder-Quotient Complements-Cycles.

We have already - again and again - noted the fact that, in the various typical cases observed and analysed by us, the two halves of the quotients (added together) give us a series of NINES. We shall now proceed a little bit further and try to see if there be any such (or similar) rule governing the remainders. For the purpose of the necessary experimentation and investigation, let us take up a more detailed consideration of the remainders obtained in each case by successive divisions of the numerator by the denominator; and let us start with $\frac{1}{7}$.

We know the successive Remainders are 3, 2, 6, 4, 6 and 1. We know already that, on reaching 6 (i.e. D ∽ N) as the remainder, half the work has been completed and that the complementary half is about to begin. Putting the above 6 figures, therefore, into two rows of three figures each, we have :

$$\begin{array}{ccc} 3 & 2 & 6 \\ 4 & 5 & 1 \\ \hline \end{array}$$

and we observe that each vertical column (of one digit from the upper row and of one from the lower one) gives us the same total (i.e. 7)!

$$\begin{array}{ccc} 3 & 2 & 6 \\ 4 & 6 & 1 \\ \hline 7 & 7 & 7 \end{array}$$

N.B. :—As our divisor is 7, it is but natural that no remainder higher than 6 is permissible i.e. that the only possible remainders are 1, 2, 3, 4, 5 and 6. (And these are the ones we actually find).

Let us now take up the cave of $\frac{1}{13}$ and note what happens. The successive remainders are 10, 9, 12, 3, 4 and 1 (the highest of which is 12). And when they are placed in two rows, we find here too, that the last three remainders are complements—from 13—of the first three remainders.

10 9 12/3 4 1

$$\begin{array}{ccc} 10 & 9 & 12 \\ 3 & 4 & 1 \\ \hline 13 & 13 & 13 \end{array}$$

In the case of $\frac{1}{17}$ the successive remainders are :

10	15	14	4	6	9	5	16/7 2 3 13 11 8 12 1
7	2	3 1	3 1	1	8 1	2	1
17	17	17	17	17	17	17	17

The last **8** remainders are thus complements—from **17--of** the first eight ones !

In the case of **19**, the Remainders are:

10	15	12	6	3	11	15	17	18
9	4	7	13	16	8	4	2	1
19	19	19	19	19	19	19	19	19

Here again the first nine remainders, when added successively to the next nine, give **19** each time.

Thus, it is clear-that, whereas the quotient-halves are uniformly complements from nine, the remainder-halves are complements from the individual divisor in each case. And this further reduces our labour in making out a list of the Remainders.

Multiples of the Basic Fractions.

Thus far, we have dealt with vulgar fractions whose numerator is unity. But what about fractions which have some other Numerator ? And the answer is: "There are several simple and easy methods by which, with a tabulated list before us of the results obtained by one or more of the processes above expounded, or even independently, we can readily put down the Recurring Decimal equivalents of the vulgar fractions of the type just under discussion.

Let us, as usual, start with $\frac{1}{7}$ and frame a chart as follows ·

$\frac{1}{7} = .\dot{1}4285\dot{7}$

$\frac{2}{7} = \cdot\dot{2}8571\dot{4}$

$\frac{3}{7} = \cdot\dot{4}2857\dot{1}$

$\frac{4}{7} = \cdot\dot{5}7142\dot{8}$

$\frac{5}{7} = \cdot 714285$; and

$\frac{6}{7} = \cdot\dot{8}5714\dot{2}$

In this chart, we observe that, in all the "proper" **fractions** having **7** as their Denominator,

(i) The same six digits are found as in the case of $\frac{1}{7}$;

(ii) they come up in the same sequence and in the same direction as in the case of $\frac{1}{7}$;

(iii) they, however, start from a different starting-point but travel in "Cyclic" order (in what is well-known as the "Clock-wise" order).

(iv) and with the aid of these rules, one can very easily obtain the **recurring** decimal equivalent of a vulgar fraction whose numerator is **higher** than **1.**

In fact, a person who is actually looking at a statement (on a board, a piece of paper, a slate etc.,) to the effect that $\frac{1}{7}=.\dot{1}4285\dot{7}$, has several easy alternative processes to choose from, for determining the decimal equivalents of all the other five possible fractions having the same denominator **(i.e. 7).** They are as follows:

The First Method.

1. The **verious** digits can be numbered and marked *in ascending order of magnitude,* thus:

(i) Unity being the least of them, the cycle for)starts with one as its starting point, travels in clock-wise cyclic order and reads:

(1)	(3)	(2)	(6)	(4)	(5)
1	4	2	8	5	7

$\dot{1}4285\dot{7}$;

(ii) 2 being the second, $\frac{2}{7}$ starts with 2 and gives **us** the answer $\cdot\dot{2}8571\dot{4}$;

(iii) There being no **3** at all, the third digit in ascending order is **4.** So $\frac{3}{7}$ begins from **4** and reads: $\cdot\dot{4}2857\dot{1}$;

(iv) The next digit (i.e. the 4th) in ascending order actually being **5, 4/7** begins with **6** and gives: $\cdot\dot{5}7142\dot{8}$;

(v) The fifth digit actually being **7, 5/7 commences** with 7 end reads:

$\cdot\dot{5}7142\dot{8}$; and

(vi) The 6th and last digit being **8,** the sixth and last fraction (i.e. $\frac{6}{7}$) starts with **8** and says:

$\cdot\dot{8}5714\dot{2}$

This is the **first** method.

Yes, but what about those cases in which the number of decimal places is more than **10** and thus, in the tabulated answer before us, some digits are found more than once?

Yes, it is perfectly true that, just as some digits are found absent [as in the case of $\frac{1}{7}$ (just seen)], there are other cases where the same digits are found more than **once.** In fact, in every case wherein the **number** of decimal places is more than **10,** this is bound to happen; and provision too there must be against it. In fact, the remedy is very simple **i.e.** that, even where digits occur more than once, there still are gradations; and, if these are taken into account, the cyclic order and the ascending order of magnitude will still operate and serve their purpose.

For example, in the case of $\frac{1}{17}$, we have **,05882......** at the very commencement; and there are two eights before us. Yes, but **88** is greater than **82**; and **therefore** we should take **82** first and **88** afterwards and do our numbering accordingly:

$\frac{1}{17}=\cdot\dot{0}$ 5 8 8 2 3 5 2 9 4 1 1 7 6 4 $\dot{7}$

(1) (10) (15) (14) (4) (6) (9) (5) (16) (7) (2) (3) (13) (11) (8) (12)

Thus, $\frac{1}{17}$ starts with zero; $\frac{2}{17}$ with **11;** $\frac{3}{17}$; with **17**; $\frac{4}{17}$ with **23**; $\frac{5}{17}$ with 29; $\frac{6}{17}$ with **3**; $\frac{7}{17}$ with **41;** $\frac{8}{17}$ with **47;** $\frac{9}{17}$ with **52;** $\frac{10}{17}$ with **58**; $\frac{11}{17}$ **with 6;** $\frac{12}{17}$ with **70;** $\frac{13}{17}$ with **76;** $\frac{14}{17}$ with **82**; $\frac{15}{17}$ with 88; **and** $\frac{16}{17}$ with **9.** The arranging in ascendmg order of magnitude has, of course, to be done carefully and correctly. But it must be admitted thst, although the procedure of counting and numbering is quite reasonable and scientifically correct, yet it is rather cumbrous, clumsy and **tiring.** Hence the need for other methods.

Yes; but what again about the cases wherein the number of digits in the decimal equivalent is much less than the denominator of the vulgar fraction in question and has thus no scope for meeting all the possible demands?

Yes, $\frac{1}{13}$ is such a case. The number of possible multiples is **12**; and the number of digits in the decimal equivalent is only **6.** (for $\frac{1}{13}=\cdot 076/923$). What is the remedy?

The remedial provision is that a multiple or two will do the trick quite satisfactorily and neatly.

Now, $\frac{1}{13}=\cdot\dot{0}7692\dot{3}$

∴ (By simple multiplication by **2**),

$\frac{2}{13}=\cdot\dot{1}5384\dot{6}$

And now, there are twelve digits in all; and these can meet the needs of all the possible multiples.

Thus—$\frac{1}{13}=\cdot\dot{0}7692\dot{3}$; and $\frac{2}{13}=\cdot\dot{1}5384\dot{6}$

∴ $\frac{3}{13}=\cdot\dot{2}3076\dot{9}$; and $\frac{4}{13}=\cdot\dot{3}0769\dot{2}$

∴ $\frac{5}{13}=\cdot\dot{3}8461\dot{5}$; and $\frac{6}{13}=\cdot\dot{4}6153\dot{8}$

∴ $\frac{7}{13}=\cdot\dot{5}3846\dot{1}$; and $\frac{8}{13}=\cdot\dot{6}1538\dot{4}$

∴ $\frac{9}{13}=\cdot\dot{6}9230\dot{7}$; and $\frac{10}{13}=\cdot\dot{7}6923\dot{0}$

∴ $\frac{11}{13}=\cdot\dot{8}4615\dot{3}$; and $\frac{12}{13}=\cdot\dot{9}2307\dot{6}$

The procedure is there and is quite correct. But, after all, one must confess that, even with this device, this counting and numbering procedure is still a cumbrous, clumsy and tiring process. Hence, let us repeat, the need for other methods.

The Second Method

The second method is one wherein we avoid even this numbering and marking etc., and (in accordance with the ***Ādyam Ādyena*** rule), multiply the opening digit or digits of the basic decimal fraction ($\cdot\dot{1}4285\dot{7}$) and determine, therefrom the starting point for the multiple in question. Thus, $\cdot\dot{1}4285\dot{7}$

∴ $\frac{1}{7}$ starts with ·14...... ... $\frac{2}{7}$ should start with ·28 etc., and (in clockwise cyclic order) give $\cdot\dot{2}8571\dot{4}$;

$\frac{3}{7}$ ought to start with ·42 etc., and give $\cdot\dot{4}2857\dot{1}$;

$\frac{4}{7}$ **must** start with **·56**; but there is no **·56** but **only** **·57** before us; and so (making allowance for a possible nay, the **actual—carrying** over of a surplus digit **from** the right leftward), we start with 67 and say: $\frac{4}{7} = \cdot\dot{5}7142\dot{8}$;

Similarly $\frac{5}{7}$ should start with **·70**; but (for the same reason as in the immediately preceding case), it actually starts with **·71** and giyes us: $\cdot\dot{7}1428\dot{5}$; and $\frac{6}{7}$ should start with **·84**, actually starts with **·85** and yields the answer: $\dot{8}5714\dot{2}$!

This is the Second method.

The Third Method

The third process is very similar; but it bases itself not on ***Ādyam Ādyena*** but on ***Antyam Antyena.*** In other words, it deals not with the opening digit but with the closing one. Thus,

∵ $\frac{1}{7}$ ends with **7**, $.\dot{1}4285\dot{7}$
∴ $\frac{2}{7}$ must end with **4** ∴ It is $\cdot\dot{2}8571\dot{4}$
∴ $\frac{3}{7}$ should end with **1** ∴ It is $\cdot\dot{4}2857\dot{1}$
∴ $\frac{4}{7}$ ought to end with **8** ∴ It is $\cdot\dot{5}7142\dot{8}$
∴ $\frac{5}{7}$ should end with **5** ∴ It is $\cdot\dot{7}1428\dot{5}$
and ∴ $\frac{6}{7}$ must end with **2** ∴ It is $\cdot\dot{8}5714\dot{2}$

This is the third method and the easiest and therefore the best of the lot.

Independent method

The above described methods are all for the utilisation of our knowledge of the decimal shape of a fraction whose numerator is unity, for deriving the corresponding decimal form of any multiple of that fraction. This is all right, so far as it goes. But what about a person who has not got such a ready-to-hand table to refer to? In such a case, should one newly prepare the basic chart and then manipulate it--cyclically—(in one of the ways just explained), for getting the required result?

That would, of course, be *absurd.* For use by such persons, we have too, a totally ***independent*** method, by which,

without resorting to any such previously prepared (or **newly** prepared) table, one can readily deal with the particular fraction on hand ! And the whole ***modus operandi*** is exactly the same as has been already explained in respect of the basic fraction **and *without the slightest difference or deviation*** in any particular whatsoever therefrom.

For example, suppose you have to decimalise $\frac{3}{7}$. Your last digit will be **1**. and as $\frac{3}{7}=\frac{21}{49}$, your ***Ekādhika Pūrva*** will be **5.** Now, go on dividing by **5,** in the usual manner; and you **get** the chart, **as explained** in the margin: $\frac{3}{7}=\frac{21}{49}=\cdot\frac{4\ 2\ 8}{1\ 4\ 2}/57\dot{1}$

After you get the **first** three digits 4; **2** and **8,** you find **that** your dividend is **28**; but this is D ∾ N (i.e. 49—21). **So** you **may** stop here and put the last three quotient-digits down as **5, 7** and **1** (the complements, from nine, of the digits already found).

Or you may continue the division till you get **21** as the dividend; and as this was your **starting-point,** you may put the **6** digits down **as** a **"Recurring"** decimal.

Thus $\frac{3}{7}=\cdot\dot{4}2857\dot{1}$

Try **this** with $\frac{7}{13}$, $\frac{13}{17}$, $\frac{5}{19}$ and so on, **with** any number of cases. And you will always **find** the same thing happening **right through all of them.** Thus, for those **who** do not have a tabulated **schedule** before them, this absolutely *independent* method is also there: and you can make full use of it.

Note :—1. In this ***independent*** method, it should also be noted that if we have to decimalise $\frac{1}{7}$, $\frac{2}{7}$, $\frac{3}{7}$, $\frac{4}{7}$, $\frac{5}{7}$, etc., we havemerely to divide **10, 20, 30, 40, 50** etc., by **7** and put down that remainder as the first remainder in each **particular** case and that: the work can be done automatically thereafter.

2. or, we may **pre-decide** the last digit in each case by taking the last digits of **7, (1) 4, (2) 1, (2) 8, 3(5), (4) 2 as the** last digits of the decimal **equivalent of** $\frac{1}{7}$, $\frac{2}{7}$, $\frac{3}{7}$, $\frac{4}{7}$, $\frac{5}{7}$ **and** $\frac{6}{7}$ **!**

Recapitulation (and Supplementation)

Over and above the ones expounded and explained herein-above, there are several other very instructive and interesting principles, features and characteristics characterising this question of the conversion of vulgar fractions into decimal ones (in respect of the remainders, the quotients etc.). For the benefit of the, students, we propose now to recapitulate, summarise, supplement and conclude this portion of the subject :

(1) As regards the remainders, we have noted that, as soon as D ∽ N comes up before us as a remainder, the remaining remainders are all complements—from the divisor (i.e. the denominator)—of the remainders already obtained ;

(2) This automatically means that the quotient—digits already obtained and the quotient--digits still to be found, are complements from Nine !

(3) If we take any remainder and multiply it by the *Caramāṅka* (the last digit), the last digit of the product is actually the quotient at that step. (The formula here is शेषाणि अंकेन चरमेण (*Śeṣaṇi* Ankena *Carameṇa*) which is therefore of the utmost significance and practical utility in mathematical computations. For instance,

(1)) The remainders are 3, 2, 6, 4, 5 and 1. Multiplied by 7 (the *Caramāṅka*) these remainders give successively 21, 14, 42, 28, 35 and 7. Ignoring the left-hand side digits, we simply put down the last dight (*Charamāṅka*) of each product ; and lo ! We get $\frac{1}{7} = \cdot\dot{1}4285\dot{7}$!

(2) $\frac{1}{13}$ The remainders are 10, 9, 12, 3, 4 and 1. Multiplied successively by 3 (the last digit), these remainders give 30, 27, 36, 9, 12 and 3. Ignoring the previous digits, we write down merely the *Caramāṅkas* the last digit) of each product ; and lo! $\frac{1}{13} = \cdot 0\dot{7}962\dot{3}$!

(3) $\frac{1}{17}$ The remainders are 10, 15, 14, 4, 6, 9, 5, 16/7, 2, 3, 13, 11, 8, 12 and 1. Multiplied by 7, they give us successively :

70, 105, 98, 28, 42, 63, 35, 112, 49, 14, 21, 91, 77 56, 84 and 7. Dropping the surplus (i.e. left-side) digits and putting down only the *Carmāṅkas* (the right-hand most digits), we have

$\frac{1}{17} = \cdot\dot{0}588235294117647\dot{}$ — ·0̇5882352/94117647̇

In fact, the position is so simple and clear that we need not multiply the whole digit, write down the product and then drop the surplus digit (or digits). We need only put down the *Charamāṅka* (the right-hand-most digit) at the very outset (as each step) and be done with it!

(4) The Geometrical-Progression = character of the Remainders gives us a clue to the internal relationship between each remainder (and its successor) (or its predecessor)! Thus, as we know one remainder, we practically know all the rest of them Thus,

In the case of $\frac{1}{7}$: As we know the first remainder is **3**, we can multiply any remainder by **3**, cast out the sevens (if any) and immediately spot out and announce the next Remainder.

$3 \times 3 = 9$; $9 - 7 = 2$; ∴ **2** is the second remainder

$2 \times 3 = 6$. This is the third rdmainder.

As 6 is D ∽ N, we may stop here and (by the rule of complements from the denominator), we may put down **4, 5** and **1** as the remaining three remainders.

Or, if we overlook the D ∽ N rule or prefer to go on with our multiplication by 3 (the Geometrical ratio), we get:

$6 \times 3 = 18$; $18 - 14 = 4$; and this is the 4th remainder.

$4 \times 3 = 12$; $12 - 7 = 5$; and this is the 5th remainder.

$5 \times 3 = 15$; $15 - 14 = 1$; and this is the 6th (and last) Remainder.

We have thus obtained from the first remainder, all the remainders:

3, 2, 6, 4, 5 and 1.

And from these, by multiplication by the *Caramāṅka* (7), we get all the 6 quotient-digits (as explained above):

1̇, 4, 2, 8, 5 and 7̇

This is not all. Instead of using the **first** remainder (3) as our Geometrical Ratio, we may take the second one **(2)**, multiply each preceding group of 2 remainders by 2 and get 32, **64** and 51 (for, by casting out the sevens, **6×2—7=5** ; and **4×2—7=1)**. And multiplying these 6 digits by 7, we again get **the** *Caramankas* 1 4 2 8 5 7 (as before).

Or we may take help from the **third** remainder **(i.e. 6)**, multiply the preceding group of 3 remainders and get 3 2 **6, 4 5 1** (for, by casting out the sevens, 3×**6—14=4 ; 2**×6—7=5 ; and **6×6—35=1.** And, multiplying these (same) 6 digits by 7, we again obtain the ***Caramāṅkas*** 142857 as before.

This procedure is, of course, equally applicable to the fourth and fifth remainders **(i.e.** 4 and 5) and can get us the same result as before This is doubtless purely academical and of no practical utility. But we are discussing a principle, nay a universally operating mathematical law and must therefore demonstrate its actual universality of application.

So, if we take the 4th remainder **(i.e.** 4) and multiply the **preceding** group of four remainders by **4,** we again get **3264/51** (For, **4×3—7=5 ; 4×2—7=1 ; 4×6—21=3 ; 4×4—14=2)** ; and the only difference is that the **first** two digits are found to have already started repeating themselves!

If we now take the 5th remainder **(i.e.** 5) and multiply the **preceding** group of **5** remainders by 5, we again get **32645/1...** (for 3×**5—14=1** ; 2x5— 7=3 ; **5×6—28=2 ; 5×4—14=6 ; 5×5—21=4 ;**

And, if we follow the same procedure with the 6th remainder **(i.e.** 1) and multiply the group of preceding remainders **by 1,** we will, of course, get **the** same preceding remainders over again !

(5) In the case of 17, the first four remainders are: **10, 15,** 14 and 4. As **4** is a manageable multiplier, we may make use of it as a convenient and suitable remainder for this purpose. Let **us** therefore multiply the group of four Remaindere (already

found) by 4 and cast out the seventeens (wherever necessary). And then we **find**:

4×10−34=6	10, **15, 14, 4**
4×15−51=9	**10, 15, 14, 4, 6**
4×14−51=5	10, **15**, 14, **4/6**, 9, 5, **16/**

4×4=16. But as D ∽ N=16, we can stop here and set down all the other remainders by subtracting each of the above digits from 17: 7, 2, 3, 13/11, 8, **12** and 1. And, multiplying each of these 16 remainders (or rather their *Caramāṅkas* i.e. units digits) by 7, we get:

$\frac{1}{17}$: $\cdot\dot{0}$ 5 8 8 2 3 5 2 /9 4 1 1 7 6 4 $\dot{7}$

Still another method

Besides (1) the corollary-Sūtra (2) each remainder × the last digit method, (3) the *Ekādhika* process from right to left and (4) the *Ekādhika* method from left to right, there is still another method whereby we can utilise the Geometrical-Progression relationship and deduce the same result by a simple and easy process. And it is this, namely, that as soon as we come across a clear ratio between one remainder (or dividend) and another, we can take that ratio for granted (as being of universal application) and work it out all through. For example,

In the case of 19, we have **10** and **5** as the first two remainders and we note that **5** is just one-half of ten. Keeping this ratio in view, we can deduce that the next remainder **should** be one-half of 5. But, as **5** is not exactly divisible by 2, we add 19 to it, make it 24 and put down its half (i.e. 12) as the next remainder. The 12 gives 6, 6 gives 3, 3(+19) gives 11,

10,	5,	12,	6,	3,	11,	**15,**	17,	**18**
9,	14,	7,	13,	16,	8,	4,	2,	1

11(+19) gives 15, 15(+19) gives 17 & (17+19) gives 18. And we stop there and put down the remaining half of the remainders by subtractions from 19. Having thus got the remainders, we multiply the *Caramāṅkas* by 1 (the last digit of the answer) and we get the quotient-digits automatically.

N.B.:—The ratio in question may be noticed at any stage of the work and made use of at any point thereof.

In the oase of $\frac{1}{17}$, we have the remainders **10** and **15** at the very start. We can make use of this ratio immediately and throughout, with the proviso that, if and when a fractional product is threatened, we can take the denominator (or as many multiples thereof as may be necessary) for making the digit on hand exactly divisible by the divisor on hand.

Thus, in the case of $\frac{1}{17}$, we have the remainders **10** and 15 to start with (the ratio being 1 to 1)). So, whenever one odd number crops up, its successor will be fractional. And we get over this difficulty in the way just explained.

And when we get a remainder which is numerically greater than the divisor, we cast off the divisor and put down the remainder. Thus,

10 gives us **15**; 15 **(+17)** gives us **48** i.e. **14**; **14** gives us **21** i.e. **4**; **4** gives us 6; 6 gives us 9; **9(+17)** gives us 39 i.e. 5; **(5+17)** gives us **33** i.e. **16**. And there we can stop.

10	15	14	4	6	9	5	16
7	2	3	13	11	8	12	1

Number of *Decimal Places*

Students generally feel puzzled and non-plussed as to how to know beforehand the number of decimal places which, on division, the decimal equivalent of a given vulgar fraction will actually consist of. In answer hereto, we must point out that, having—in the immediately preceding sub-section on this subject—made a detailed, analytical study of the successive remainders, we have, in every case before us, practically a tabulated statement from which (without actual division to the very end) we can postulate beforehand all the forthcoming remainders. And the tabulated statement has the further merit that it can be prepared, at any time, at a moment's notice!

All this means, in effect, that,

(i) As soon as 1 (or other starting point) is reached (in our mental analysis), we will have completed the whole work of-decimalisation and therefore know the actual number of decimal places coming ahead. The rases $\frac{1}{7}$, $\frac{1}{13}$, $\frac{1}{17}$, $\frac{1}{19}$, etc., have all proved this.

(ii) As soon as we reach the difference between the numerator and denominator, we know we have done half the work and that the other half is yet to come. The cases of $\frac{1}{7}$ etc., (which we have dealt with in *extenso*) have proved this too.

(iii) As soon as we reach a fairly small and manageable remainder (in our mental calculation), we know how many more steps we should expect.

Let us again take the case of $\frac{1}{7}$ by way of illustration. The first remainder is 3; and used as a successive multiplier (with the provision for the casting out of the sevens), that first remainder—multiplicr brings us on to 1.

When we have done two steps and got 1 and 4 as the first two quotient-digits, we find 2 is the remainder. Multiplying the first group of two digits (14) by 2, we get 28 as the second-group (with the remainder also doubled i.e. $2\times2=4$). 14/28/.

Multiplying 28 by 2, we get $28\times2=56$ as the third group and $4\times2=8$ as the remainder. And then, by casting out the sevens, we obtain 57 as the quotient-group and 1 as the remainder! And as this was our starting-point, we stop further computations and decide that $\frac{1}{7}$, when decimalised, has 6 decimal places in the answer.

Going hack to the case of $\frac{1}{17}$, the student will remember that, after 4 steps, we got.0588 as the quotient-digits and 4 as the remainder. Multiplying the former by the latter, we obtained 2353 as the second quotient-group and $4\times4=16$; as the remainder; and there we stopped, (because we had the first 8 digits on hand and knew the other 8 digits). Thus $\frac{1}{17}$ gave us 16 digits.

(As a Geometrical series is of the atandard form 1, r, r^2 and so on, we are able to utilise 2 and 2^2 (in the case of $\frac{1}{7}$), 4 and 4^2 (in the case of $\frac{1}{17}$) and so on for helping us to pre-determine the number of decimal places in the answer. This is the Algebraical principle utiliied herein.

Note :—1. We need hardly point out that the *Ekādhika* method has the supreme and superlative merit of lightening our division (and multiplication) work. For instance, in the case of $\frac{1}{19}$, $\frac{1}{29}$ etc., we have to do our division-work, at stage after stage, by successive division, not by 19 or 29 etc., (the original denominator) but by 2 or 3 etc. (the *Ekādhika-pūrva*). And this is the case with regard to every case i.e. that we perform all our operations—in this system—with much smaller divisors, multipliers eto., and this rule is invariable. What a tremendous saving in effort, labour, time and cost!

2. We have purposely treated this subject at great length and in elaborate detail, because it is very essential that the whole matter should be clearly understood, thoroughly assimilated and closely followed so that, even without the help of a teacher, the student may be enabled to work out these methods independently in other similar cases and to *know*—with absolute certainty—that ANY and EVERY vulgar fraction can be. readily tackled and converted into the corresponding recurring decimal (*whatever* may be the complexity thereof and the number of decimal places therein). In fact, in as much as these simple and easy processes are available—and suitable—for ALL possible denominators and for all possible numerators, the decimal (and especially the recurring decimal) should no longer be a bugbear to the student. On the contrary, they should be the most welcome of all welcome friends!

Some Characteristic features (General and Special)

(1) In the cases of fractions with *prime numbers* (like 19, 29, 59 etc.,) as denominators, the maximum number of decimal places is one less than the denominator! This is self-evident and requires no elaboration.

(2) *Usually,* it, or a sub-multiple thereof, is the actual number.

(3) *Generally,* the rule of complements (from nine) is found in operation amongst them.

(4) For fractions like $\frac{1}{21}$, $\frac{1}{49}$, $\frac{1}{63}$ etc., (where the denominator are products of prime numbers), the number of digits depends on the various respective factors in each case (as will be presently elucidated).

(5) If and when the decimal-fraction obtained from one of the factors of the denominator is exactly divisible by the other factor (or factors), the division by the second factor leaves no remainder. And therefore the number of decimals obtained by the first factor is not added to! Thus,

(i) $\frac{1}{7\times3}=\frac{\cdot\dot{1}4285\dot{7}}{3}$

Here, the numerator on the R. H. S. being exactly divisible by 3, it divides out and haves no remainder. Therefore, the number of digits continues the same.

This means that, in every case wherein the complementary halves (from nine) are found, the numerator on the R H S *must* necessarily be divisible by 3, 9 etc. And by multiplying the denominator in such a case by such factors, we cause no difference to the number of decimal places in the answer. And consequently, we have:

$$\frac{1}{63}=\frac{1}{7\times9}=\frac{\cdot142857}{9}=\cdot\dot{0}1587\dot{3}$$; and so on.

Going back to the *Ekanyūna Sūtra* (as explained in connection with the Sanskrit Alphabetical code), we know that $142857=143\times999=11\times13\times3^3\times37$. This means that since the numerator is divisible by *11.13,* 3, 9, 27, 37, 33, 39, 99, 117, 297, 351 and 999, the multiplication of the Denominator (7)

by anyone of these factors will make no difference to the number of decimal places in the answer.

(ii) $\frac{1}{39}=\frac{1}{13\times3}=\frac{\cdot076923}{3}$

Here too, all the above considerations apply. And, since $76923=77\times999=3^3\times7\times11\times37$, therefore these factors (and combinations of factors) will, by multiplying the denominator, make no difference to the number of decimal places. (Note. $999999=999\times1001=999\times7\times11\times13$).

(6) $\frac{1}{69}=\frac{1}{23\times3}$ and comes under the same category (with 22 digits in the answer (just like $\frac{1}{23}$).

(7) $\frac{1}{49}$ is a special case and stands by itself. Naturally it should have been expected to provide for 48 places. But, as a matter of fact, it gives only 42; and for a perfectly valid and cogent reason i.e. that, out of the 48 possible multiples, six (i.e. $\frac{7}{49}$, $\frac{14}{49}$, $\frac{21}{49}$, $\frac{28}{49}$, $\frac{35}{49}$ and $\frac{42}{49}$) go into a different family, as it were-and take shape as 3, $\frac{2}{7}$, $\frac{3}{7}$, $\frac{4}{7}$, $\frac{5}{7}$ and $\frac{6}{7}$; have their places there as $.\dot{1}4285\dot{7}$, $\cdot\dot{2}8571\dot{4}$ and so forth and need no place in the $\frac{1}{49}$ etc., group! And thus, since 6 go out of the 48, the remaining 42 account for the 42 places actually found in the decimal equivalent of $\frac{1}{49}$! This is not a poet's mere poetic phantasy but a veritable mathematical verity!

(8) $\frac{1}{79}$ is, in a way, an exception, as it contains only 13 digits. And, as this is an odd number, the question of the two complementary halves does not arise! 13, however, *is* a sub-multiple of 78; and there is no deviation from the normal in this respect. An at-sight-one-line mental method will soon be given for $\frac{1}{79}$ (in this very chapter).

(9) Similarly $\frac{1}{89}$ has 44 digits and thus conforms to the sub-multiple rule. And this implies that, like $=,it will need another complete turn of the wheel (in one of its multiples) in order to meet the needs of all the multiples! (An *incredibly easy* method will be shown in this very chapter for reeling off the answer in this case).

(10) $\frac{1}{99}$ has only two recurring places ($\cdot\dot{0}\dot{1}$); but the whole *gamut* can be and has been provided for, therewith.

(11) In the case of basic fractions ending in 3, the denominator is first multiplied by 3 and gives us the *Ekādhika,* and the last digit in the answer is also 3.

(12) $\frac{1}{33}$ (like $\frac{1}{99}$) has only two decimal places ($\cdot\dot{0}\dot{3}$).

(13) $\frac{1}{43}$ has only 21 digits. 21 is a sub-multiple of 42 but is odd and gives no scope for the complementary halves.

(14) $\frac{1}{63}$ has only 13 digits (a sub-multiple but odd)

(15) $\frac{1}{83}$ has only 41 digits (similarly).

(16) $\frac{1}{73}$ is special. Since $73\times137=10001$ and since

$$10001\times9999=99999999 \therefore \frac{1}{73}=\frac{137}{10001}=\frac{137\times9999}{99999999}$$

$$=\cdot\dot{0}136/986\dot{3}$$ (by *Ekanyūna Sūtra*)

(17) And, conversely, $\frac{1}{137}=\frac{73}{10001}=\frac{73\times9999}{99999999}=\cdot\dot{0}072992\dot{7}$

(18) $\frac{1}{93}$ will be discussed a little later.

(19) In the case of fractions whose denominators end in 7, the last digit is also 7; and the *Ekādhika* is obtained from the denominator multiplied by 7.

(20) $\frac{1}{7}$ arid $\frac{1}{17}$ have been dealt with in detail already.

(21) $\frac{1}{27}$ and $\frac{1}{37}$ are special (because $27\times37=999$). And their decimal forms are. $\cdot\dot{0}3\dot{7}$ and $\cdot\dot{0}2\dot{7}$.

(22) $\frac{1}{47}$ has 46 digits.

(23) $\frac{1}{57}=\frac{1}{19\times3}$ and has only 18 digits

(24) $\frac{1}{67}$ has 33 digits (odd)

(25) $\frac{1}{77}$ has been discussed already (number $\frac{1}{7}$)

(26) $\frac{1}{87}=\frac{1}{29\times3}$ and has 28 digits.

(27) $\frac{1}{97}$ has its full quota of 96 digits.

(28) In the case of fractions with denominators ending in 1, the *Ekādhika* comes from the denominator multiplied by 9: and the last digit is 9.

(29) $\frac{1}{11} = \cdot\dot{0}\dot{9}$

(30) $\frac{1}{21} = \frac{1}{7 \times 3}$ and has been discussed under 7.

(31) $\frac{1}{31}$ will come up a little later.

(32) $\frac{1}{41}$ is special $\because 41 \times 271 = 11111$

$$\therefore \frac{1}{41} = \frac{271}{11111} = \frac{271 \times 9}{99999} = \cdot\dot{0}243\dot{9} \text{ (odd)}$$

(33) And, conversely, $\frac{1}{271} = \frac{41}{11111} = \frac{41 \times 9}{99999} = \frac{00369}{99999} = \cdot\dot{0}036\dot{9}$

(34) $\frac{1}{51} = \frac{1}{17 \times 3}$ and has 16 digits.

(35) $\frac{1}{61}$ has 60 digits.

(36) $\frac{1}{71}$ has 35 digits (odd)

(37) $\frac{1}{81} = \frac{1}{27 \times 3} = \frac{\dot{0}3\dot{7}}{3} = \cdot\dot{0}1234567\dot{9}$ (a very interesting number).

(38) $\frac{1}{91} = \frac{1}{13 \times 7}$ and has already been discussed under 7 and under 13. And besides,

$$\frac{1}{91} = \frac{11}{1001} = \frac{11 \times 999}{999999} = \cdot\dot{0}10/98\dot{9}$$

But

But here a big BUT butts in and exclaims: "Yes; all this is all right in its own way and so far as it goes. But, as our denominators go on increasing, we note that, although the last digit of the decimal fraction is 1,3,7, or at the most 9 and no more, yet, the *Ekādhika Pūrva* goes on increasing steadily all the time and we have to multiply or divide successively by bigger and bigger *Ekādhikas,* until, at last, with only two-digit denominators like 61, 71 and 81 and so on, we have now to deal with 55, 64, 73 etc., as our multipliers and divisors, and surely this is ***not*** such an easy process.

The objection is unobjectionable; nay, it is perfectly correct. But we meet it with quite a variety of sound and valid answers which will be found very cogent and reasonable. They are as follows:

(i) Even the biggest of our *Ekādhikas* are nowhere—in respect of bigness—near the original divisor. In every case, they are smaller. But this is only a theoretical and dialectical answer from the *comparative* standpoint and does not really meet the intrinsic objection (about the Vedic methods being not only *relatively* better but also being free from all such flaws altogether)! We therefore go on and give a satisfactory answer from the positive and construotive stand-point.

(ii) Even though the *Ekādhika* is found to be increasingly unmanageably big, yet the remainders give us a simple and easy device for getting over this difficulty. This we shall demonstrate presently.

(iii) The *Ekādhika* (so far explained and applied) is not the whole armoury. There are other Auxiliaries too, wherein no such difficulty can crop up. These we shall expound and explain in a subsequent, but sufficiently near chapter of this very volume; and they will be found capable of solving the problem *in toto*; and

(iv) Above all, there is the CROWNING GEM of all coming up in a near chapter and unfolding before our eyes a formula whereby, however big the denominator may be, we can—by mere mental one-line Vedic arithmetic—read off the quotient and the remainder, digit by digit! This process of "Straight Division", we have already referred to and shall explain and demonstrate, in a later chapter, under this very caption "Straight (or Instantaneous) Division".

In the meantime, just now, we take up and explain the way in which the remainders come to our rescue and solve this particular problem for us.

Let us take first, the cae of $\frac{1}{23}$. We know immediately that the last digit of the decimal is **3** and that the *Ekādhika* is **7.** And then we work as follows:

(i) Multiplying digit after digit (as usual) **by 7,** we have:

$\frac{1}{23}=\frac{3}{69}=$ ·04347826086 /9 5 6 5 2 1 7 3 9 1 3
6 /3 4 3 1 1 5 2 6 2

or (ii) dividing digit by digit (as usual) by **7,** we have:

$\frac{1}{23}=\frac{3}{69}=$ ·0 4 3 4 7 8 2 6 0 8 6/9 5 6 5 2 1 7 3 9 1 3
2 3 5 5 1 4 6 4 6

These are the usual *Ekādhika Pūrva* methods. But

(iii) we observe in the first chart, after two digits **(1 & 3** have been obtained), the next leftward group **(39)** is exactly three times (the extreme-right-end one) and we can immediately profit by it. Thus **39** gives us **117,** out of which we put down **17** and keep **1** to carry over; **17** gives us 51+1=52. **52** gives us **156,** out of which we set down **56** and keep **1** to carry over. **56** gives us 168+1=169. of these, we put **69** down and keep 1 to be carried over; and so on. In fact, the whole procedure is exactly like the one which we followed from left to right in respect of $\frac{1}{49}$ (=·020408 **16 32......**). Thus we have:

$\frac{1}{23}=\frac{3}{69}=\dot{0}434782608$, **69, 56, 52, 17, 39,** $1\dot{3}$

or (iv) if we wish to start from the left end, go on to the right, that too is easy enough.

We note that, the first digits being completed, we get 8 as the Remainder. We can immediately work out this process by multiplying each two-digit group by 8 (as we did in the case of $\frac{1}{49}$ by 2) and frame the following chart:

·04	: 32	: 72	: 24	: 08	: 64	: and so on
	: 2	: 6	: 2	:	: 5	:
	: 34	: 78	: 26	:	: 69	:

These multiplications by **3** to the left and by **8** to the right are easy enough. Aren't they?

Let us now take up and try $\frac{1}{47}$ (as promised at an earlier stage). Obviously, the last digit is **7** and the *Ekādhika* is **33.** This is rather unwieldy as a multiplier or as divisor. We should therefore try and see what we can get from the Remainders. We find them to be **10, 6** etc. We can immediately pounce up on this **6** for our purpose and work in this way:

·02 being the first two digits of the quotient and **6** being our ratio, the next two digits are obviously 12. These $\times$ 6 should give us **72 ;** but as **4** will be coming over from the right, we add the **4** and put down **76.** This should give us **456,** of which the first digit has already been taken over to the left. So **56** remains. But this will be increased by **3** (coming from the right) and will become **59**—This gives us **57, 44** and **68** for the next three **2**-digit groups and 08 for the one thereafter. The **08** group of two digits gives us **48** which, with the carried digit becomes **51.** This gives us **06** and **36** (which becomes **38).** And then we have **28** (turning into **29,** then **74** which becomes **78)** and so forth.

Thus we have : $\frac{1}{47}$ = ·02 **12 76 59 44 68 08 51 382/**
97 8

Here we notice that, exactly after **23** digits, the complements (from nine) have begun. So, we can complete the second half and say : $\frac{1}{47}$ = $\dot{0}$2 **12 76 59 57 44 68 08 51 06 382**
97 87 23 40 42 55 31 91 48 93 61$\dot{7}$

We have thus avoided the complicated divisions by the original divisor **47** and also the divisions and multiplications by the unmanageable *Ekādhika* **33 ;** and, with the easy remainder **6** as our multiplier, we have been able to obtain all the **46** digits of the answer !

This merely shows that these are not cut-and-dried mechanical processes but only rules capable of being applied to the special kind of cases which they are particularly designed to meet and fit into.

And, as for a cut-and-dried formula capable of universal application, that too is forthcoming (as already indicated) and will be dealt with, very soon.

Let us now take up $\frac{1}{31}$ which, a little earlier, we promised to deal with soon afterwards. In this case, the last digit is 9; and the *Ekādhika* is 28 (which is *nearly* as big as the original denominator itself)! We should therefore sift the remainders and find a suitable *auxiliary* therefrom.

In this case, we find **7** is the first significant remainder. So, leaving the *Ekādhika* process out of account for the moment, we may use the Geometrical Progression principle and achieve our purpose thereby (as we did-with 6-in the case of $\frac{1}{47}$). But let us proceed further and see whether a *still more easily manageable* remainder is available further up.

Well; we observe:

$\frac{1}{31}$ = ·032258 (with remainder 2). The actual remainders (in order) are: 10, **7**, 8, 18, **25** and **2**!

This suits us most admirably, and we proceed further with the help of 2. Thus:

·032258/064516/129,032/258064......

But this means that, after only **15** digits (an odd number), the decimal has already begun to *recur*! So, we simply say:

$\frac{1}{31} = \cdot\dot{0}3225806451612\dot{9}$!

What a simple and easy device!

Let us now take up $\frac{1}{97}$. The last digit is 7; but the *Ekādhika* will bo 68! So, we seek help from the Remainders. They are: 10, 3 etc., and the quotient-digits are ·0103...

So, multiplying each quotient-group (of 2 digits each) by 3 (as we did, by 2, in the case of $\frac{1}{49}$), we get:

$\frac{1}{97}$ = ·010309 **27** 81
                                  2
                                  ---
                                  83 etc. etc!

Let us take one more example (i.e. $\frac{1}{697}$) and conclude. The last digit is 7; but the *Ekādhika* will be 698! It will surely not be an enviable task for even the most practised and experienced statistician to multiply or divide, at each step, by such a big figure! We therefore again seek help from the remainders and the Geometrical Progression Rule.

The Quotient-digits are ·001 etc., and the successive remainders are 10, 100, 3 etc.! This means that we should multiply each group of three quotient-digits by 3 and get our answer (to any number of decimal-places). We thus have:

$\frac{1}{997}$ = ·001 : 003 : 009 : 027 : 081 : **243** : **729**
: 3
732 etc. eto.

The Converse operation.

Having dealt, *in extenso,* with the conversion of vulgar fractions into their equivalent recurring decimals, we now take up the CONVERSE process i.e. the conversion of decimals into the equivalent vulgar fractions. We do not, however, propose to go into such a detailed and exhaustive analytical study thereof (as we have done in the other case) but only to point out and' explain *one* particular principle, which will be found very useful in this particular operation and in many subsequent ones.

Tho principle is based, on the simple proposition that $\cdot\dot{9}=\frac{9}{9}=1$; $\cdot\dot{9}\dot{9}=\frac{99}{99}=1$; $\cdot\dot{9}9\dot{9}=\frac{999}{999}=1$; and so forth ad *infinitum.* It therefore follows that all recurring decimals whose digits are all nines are *ipso facto* equal to unity; and if a given decimal can be multiplied by a multiplier in such a manner as to produce a product consisting of only nines as its digits, the operation desired becomes automatically complete.

For instance, let us first start with the now familiar decimal $\cdot\dot{0}7692\dot{3}$. In order to get 9 as the last digit, we should multiply this by 3. Setting this product down (·230769), we find that, in order to get 9 as the penultimate digit, we should add 3 to the 6 already there. And, in order to get that 3, we should multiply the given multiplicand by 1. On doing this, we find that the totals (of the two rows) are all nines! So we stop there and argue that, because the given decimal×13=999999 (i.e. 1), therefore the fraction should be $\frac{1}{13}$. In fact, it is like saying 13x=1 ∴ x=$\frac{1}{13}$

```
 ·0 7 6 9 2 3
           1 3
 ------------
 ·2 3 0 7 6 9
 0·7 6 9 2 3
 ------------
 ·9 9 9 9 9 9 =1
```

(2) Secondly, let us take the 'case of $\cdot\dot{0}3\dot{7}$ and see how this works. Here as the last digit is 7, so, in order to get 9 as the last digit of the product, we should multiply it by 7. And, putting 259 down, we should add 4 to obtain 9 as' the penultimate digit. And, in order to get that 4 there, we should multiply the multiplicand by 2. And, on doing so, we find that the product is $\dot{9}9\dot{9}$ (=1). Therefore, the fraction $\times 27 = 1 \therefore x = \frac{1}{27}$

$$\begin{array}{r} \cdot\dot{0}\;\;\;\; \\ 2\;7 \\ \hline 2\;5\;9 \\ 0\;7\;4\;\;\; \\ \hline 0\cdot\dot{9}\;9\;\dot{9} = 1 \end{array}$$

(3) When we try the case of $\dot{1}4285\dot{7}$; we find that multiplication by 7 gives us the all-nine product. $\cdot\dot{9}9999\dot{9}$ (=1) ; and therefore we say $\cdot\dot{1}4285\dot{7} = \frac{1}{7}$

(4) $\dot{0}4761\dot{9}$. We first multiply by 1, see 1 in the penultimate place, have to add 8 thereto, multiply by 2 (for getting that 8) and thus find that the required answer is $\frac{1}{21}$.

$$\begin{array}{r} \cdot\dot{0}\;4\;7\;6\;1\;9 \\ 2\;1 \\ \hline \cdot 0\;\;\;\;\;\;\;\;\;\;\; \\ 0'9\;5\;2\;3\;8\;\;\; \\ \hline \cdot\dot{9}\;9\;9\;9\;9\;9 = 1 \end{array}$$

(5) Similarly, we may take up various other decimals (including the long big ones like the equivalents of $\frac{1}{17}$, $\frac{1}{27}$, $\frac{1}{47}$, $\frac{1}{83}$, $\frac{1}{117}$, $\frac{1}{17}$, $\frac{1}{19}$, $\frac{1}{23}$, $\frac{1}{89}$ etc., and invariably we find our purpose achieved.

(6) But, what about decimals ending in even numbers or 5? Well; no integral multiplier can possibly get us 9 as the last digit in the product. And what we do in such a case is to divide off by the powers of 2 and 6 involved and use this new method with the final quotient thus obtained. Thus, if we have to deal with 285714 we divide it off by 2, get $\cdot\dot{1}4285\dot{7}$ as the quotient and find that multiplication thereof by 7 gives us the product $\cdot\dot{9}9999\dot{9} = 1$. And therefore we say :

$$\begin{array}{r} \cdot 2)\cdot\dot{2}8571\dot{4} \\ \hline \cdot 142857 \end{array}$$

$$\frac{x}{2} \times 7 = 1 \therefore x = \frac{2}{7}$$

(7) Let us now try the interesting decimal. 012345679. On applying this new method, we find that multiplication by 81 gives us 1 as the product $\therefore x = \frac{1}{81}$

$$\begin{array}{r} \cdot\dot{0}\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\; \\ 8\;1 \\ \hline \cdot 0\;1\;2\;3\;4\;5\;6\;7\;9 \\ '9\;8\;7\;6\;5\;4\;3\;2\;\;\; \\ \hline \cdot\dot{9}\;9\;9\;9\;9\;9\;9\;9\;\dot{9} = 1 \end{array}$$

N.B. 1. The student should also make use of the *Ekanyūna* formula. This is readily applicable in every case of "Complementary halves" (including $\frac{1}{7}$, $\frac{1}{13}$, $\frac{1}{17}$, $\frac{1}{19}$ etc.

Thus. $\dot{.}14285\dot{7} = \frac{143 \times 999}{999\ \ 999} = \frac{11 \times 13}{1001} = \frac{11 \times 13}{11 \times 13 \times 7} = \frac{1}{7}$

Similarly. $.076923 = \frac{77 \times 999}{1001 \times 999} = \frac{1}{13}$; and so on

2. Similarly, with regard to other factors too, it goes without saying that the removal, in general. of common factors (from the decimal and the denominator) facilitates and expedites the work.

3. The subsequent chapters on "AUXILIARY FRACTIONS" and "DIVISIBILITY" etc., will expound and explain certain very simple and easy processes by which this work (of arithmetical factorisation) can bc rendered splendidly simple and easy; and

4 Above all, the forthcoming "Straight Division" method will not merely render the whole thing simple and easy but also turn it into a pleasure and a delight (even to the children)

Some Salient Points and Additional Traits.

Thus, the *Ekādhika* process (forwards and backwards) and the Geometrical Progression relationship between the remainders have given us the following three main principles:

(i) The quotient-complements (from 9);

(ii) The remainder-complements (from the Denominator);

& (iii) The multiplication of the *Caramāṅka* (last digit) of the remainders by the *Caramāṅka* ((thelast digit) of the decimal, for obtaining each digit of the quotient.

Now, *apropos* of and in connection with this fact, the following few important and additional traits should also be observed and will be found interesting and helpful:

(I) In the case of $\frac{1}{19}$, theremainders are (1), 10, 5, 12, 6, 3, 11, 15, 17, 18/9, 14, 7, 13, 16, 8, 4, 2, 1 and the quotient digits are: 0 5 2 6 3 1 5 7 8/9 4 7 3 6 8 4 2 1 .

(i) Each remainder by itself, if even; and with the addition of the denominator, if odd, is double the next remainder. This follows from the *Ekādhika* being 2.

(ii) Each quotient-digit is the last digit of its corresponding remainder. This is because 1 is the last digit of the decimal.

(2) In the case of,',, the remainders are: (1) 10, 13, 14, 24, 8, 22, 17, 25, 18, 6, 2, 0, 26, 28/19, 16, 15; 5, 21, 7, 12, 4, 11, 23, 27, 9, 3, and 1.

(i) The quotient-digits are the last digits thereof (for the same reason as above)

(ii) Each remainder (by itself or in conjunction with the denominator or double of it)=thrice its successor; and

(iii) Each remainder *plus* ita suocessor's successor=the next remainder thereafter. Thus 10+14=24; 13+24−29=8; 14+8=22; 24+22−29=17; 8+17=25; 22+25−29=18; 17+18−29=6;

25+6−29=2; 18+2=20; 6+20=26; and so on!

N.B. Note the casting off of the denominator all through.

(3) In the case of $\frac{1}{89}$, the quotient-digits are:

0 112 3595 5056 1797 752 808
9887 6404 4943 8202 247 191̇

and the remainders are (1), 10, 11, 21, 32, 53, 85, 49, 45, 5, 50, 55, 16, 71, 87, 69, 67, 47, 25, 72, 8, 80, 88, 79, 78, 68, 57, 36, 4, 40, 44, 84, 39, 34, 73, 18, 2, 20, 22, 42, 64, 17, 81, 9 and 1. (Note the Ratio 9 : 1.)

The remarkaole thing here is that the numerator+the first remainder=the second remainder and that all through, the sum of any two consecutive remainders is the next remainder thereafter! Thus 1+10=11; 10+11=21; 11+21=32; and so on to the very end.

The general form herefor is a, d, a+d, a+2d, 2a+3d, 3a+5d, 5a+8d etc. The student who knows this secret relationship (between each remainder and its successor) can reel the 44 digits of the answer off, at sight, by simple addition!

(4) In the case of $\frac{1}{79}$, the remainders are:

(1) 10, 21, 52, 46, 65, 18, 22, 62, 67, 38, 64, 8 and 1.

The general form herefor is a, d, a+2d, 2a+5d etc. Knowledge of this relationship will be of splendid practical utility in this case. (Note the Ratio 8 : 1.)

(5) In the case of $\frac{1}{89}$ the remainders are:

(1) 10, 31, 34, 64, 19, 52, 37, 25, 43, 16, 22, 13, 61, 58, 28, 4, 40, 55, 67, 49, 7 and 1. (Note the Ratio 7 : 1.)

The general form herefor is obviously a, d, a+3d, 3a+10 d, 10a+33d etc.

(6) In the case of 69, the remainders are:

(1) 10, 41, 56, 29, 54, 9, 31, 15, 32, 25, 14, 22, 43, 17, 62, 48, 8, 21, 33, 35, 55, 19, 13, 12, 2, 20, 23, 53, 58/49, 18, 3, 30, 5, 50, 28, 44, 27, 34, 45, 37, 16, 42, 7, 11, 51, 38, 26, 24, 4, 40, 40, 47, 57, 39, 36, 6 and 1 (Note the Ratio 6 : 1).

Here the general form is a, d, a+4d, 4a+17d etc.

Inductive conclusion

Having thus examined the cases of $\frac{1}{59}$, $\frac{1}{79}$, $\frac{1}{89}$ and $\frac{1}{69}$, we note tho following:

(i) In every case, we start with 1 (the basic numerator) as a sort of pre-natal remainder (which is perfectly justified because we are dealing with a recurring decimal), and we call it a;

(ii) In every case, the first actual remainder is 10; and we call it d;

(iii) And then the successive remainders are a, d, a+d, a+2d, a+3d, a+4d respectively (wherein the coefficient of x is obviously the deficit of the penultimate digit from 9!

Thus for $\frac{1}{89}$, we have a+1d ;
for $\frac{1}{79}$, we have a+2d ;
for $\frac{1}{69}$, we have a+3d ;
for $\frac{1}{59}$, we have a+4d ; and so on.

(iv) And this relationship is maintained **systematically** all through. In other words, each **remainder+the** next one or double that or thrice that **etc.=the further** subsequent remainder. Arguing thus, let us try $\frac{1}{39}$ As **3** is 6 less than 9, ∴ the general form should be **a+6d**. This means 1, 10, 61 (i.e. 9), 64 (i.e. 12), 3, 30 (i.e. 4) and 27 (i.e. 1). And we find this to be actually correct.

(v) And, in case the penultimate digit is more than 9, we should react by subtracting d (and not add to it) at the rate of 1 for each surplus. Thus, our chart will now read-a, d, a—d, d- (a—d) i.e. 2d—a, and so on. For instance, for $\frac{1}{109}$, the remainders will be (1) 10, **—9,** 19,—28, 47 and so on.

(vi) And, over and above all these details which are different for different numbers (as explained above), there is **one multiplier** (namely **10**) which is applicable to all cases! And thus, whatever fraction we may be dealing with, 2, 4, 5, 8 or any remainder whatsoever can be safely put in into the next place with a zero added! The student will observe that, in *all* the examples dealt with hereinabove (not only in this particular subsection), every such remainder of twn digits (ending in zero) has been invariably preceded hy the same number (without the zero)!

With the help of this rule applicable in all cases and the special rules (shout d, 2d, 3d, 4d etc.,) enjoined for the different individual cases, the student should easily now bc in a position to make a list of the successive remainders in each case and therefrom, by *Caramāṅka* multiplication, put down the successive quotient-digits without further special labour!

These and many more interesting features there are in the Vedic decimal system, which can turn mathematics for the children, from its present excruciatingly painful character to the exhilaratingly pleasant and even funny and delightful character it really bears!

We have, however, already gone into very great details; and this chapter has already become very long. We therefore conclude this chapter here and hold the other things over for a later stage in the student's progress.

GTRAIGHT DIVISION

We now go on, at last, to the long-promised **Vedic process** of STRAIGHT (AT-SIGHT) DIVISION which is a simple and easy application of the *URDHVA-TIRYAK* Sūtra which is capable of immediate application to *all* cases and which we have repeatedly been describing as the "CROWNING GEM of all" for the very simple reason that over and above the universality of its application, it is the most supreme and superlative manifestation of the Vedic ideal of the at-sight mental-one-line method of mathematical computation.

Connecting *Link*

In order to obtain a correct idea of the background, let us go back, very briefly, for a very short while, to the methods which we employed in the earlier chapters on division; and let us start with the case of $\frac{38982}{73}$.

According to the first method (under the *Nikhilam* etc., Sūtra), our chart will read as follows:

73	3 8 9	8 2	
$\overline{27}$	6 21		
	28	98	∴ Q=534
		116406	
			and R = 0
	3 14 58	($\overline{26}$) 28	
	534	(36) 10	

According to the second method (by *Parāvartya* formula), we say:

73	3 8 9	8 2
$\overline{27}$	9 —9	
$\overline{3}\overline{3}$	51	—51
		153 —153
	3 17 51	23 949
	5 3 4	0 0

We have felt, and still feel, that even these comparatively short, intellectual and interesting methods are cumbrons and clumsy (from the Idealistic Vedic standpoint). And hence the clamant need for a method which in free from all such flaws and

which fulfils the highest Idealistic ideal of the Vedic Sūtras.

And that is as follows:

Out of the divisor 73, we put down only the first digit (i.e. 7) in the Divisor-column and put the other digit (i.e. 3) "on top of the *flag*" (by the ***Dhvajāṅka Sūtra***), as shown in the chart alongside.

3 :	38	9	8 :	2 :	
7 :		3	3 :	1 :	
:	5	3	4 :	0 :	

The entire division is to be by **7**; and the procedure is as explained below:

As one digit has been put on top, we allot one place (at the right end of the dividend) to the remainder portion of the answer and mark it off from the digits by a vertical line.

(i) We divide **38** by 7 and get **5**, as the quotient and **3** as the remainder. We put **5** down as the first quotient-digit and just prefix the remainder (3) up before the 9 of the dividend. In other words, our actual second-step Gross Dividend is **39.** Prom this, we, however, deduct the product of the indexed **3** and the first quotient-dight **5** (i.e. 3 X 5=15). The remainder **(24)** is our actual Nett-Dividend. It is then divided by **7** and gives us **3** as the second quotient-digit and **3** as the remainder, to be placed in their respective places (as was done in the first step). From **38** (the gross dividend thus formed), we subtract **3**×the second quotient-digit 3 i.e. **9,** get the remainder **29** as our next actual dividend and divide that by **7**. We get 4 as the quotient and **1** as the remainder. This means our next Gross Dividend is **12** from which, as before, we deduct **3**×the third quotient-digit 4 (i.e. **12**) and obtain 0 as the remainder. Thus we say: Q is 534 and R is zero. And this finishes the whole procedure; and all of it is one-iine mental Arithmetic (in which all the actual Division is done by the simple-digit Divisor 7)!

The *Algebraical Proof* hereof is very simple and is based on the very elementary fact that all arithmetical numbers are

merely Algebraical expressions wherein x stands for ten. For instance, $3x^2+5x+1$ is merely the algebraical general expression of which (with x standing for 10) the arithmetical value is 351.

Remembering this, let us try to understand the steps by means of which 3 8 9 8 2 is sought to be divided by 73. Algebraically put (with x standing for 10), this dividend is $38x^3+9x^2+8x+2$; and this divisor is $7x+3$. Now, let us proceed with the division in the usual manner.

$$
\begin{array}{l}
7x+3\ )\ 38x^3+9x^2+8x+2\ (5x^2+3x+4 \\
\quad 35x^3+15x^2 \\
\hline
\quad 3x^3-6x^2 \\
\quad =24x^2+8x \\
\quad 21x^2+9x \\
\hline
\quad 3x^2-x \\
\quad =29x+2 \\
\quad 28x+12 \\
\hline
\quad x-10 \\
\hline
\quad =10-10=0 \\
\hline
\end{array}
$$

When we try to divide $38x^3$ by $7x$, our first quotient-digit is $5x^2$; and, in the first step of the multiplication of the divisor by $5x^2$, we get the product $35x^3+16x^2$; and this gives us the remainder $3x^3+9x^2-15x^2$, which really means $30x^2+9x^2-15x^a=24x^2$. This plus $8x$ being our second-step dividend, we multiply the divisor by the second quotient-digit $3x$ and subtract the product $21x^2+9x$ therefrom and thus get $3x2-x$ as the remainder. But this $3x^2$ is really equal to $30x$ which (with $-x+2$) gives us $29x+2$ as the last-step dividend. Again multiplying the divisor by 4, we get the product $28x+12$; and subtract this $28x+12$, thereby getting $x-10$ as the Remainder. But x being 10, this remainder vanishes! And there you have the whole thing in a nut-shell.

It wiil be noted that the arithmetical example just hereabove dealt with (i.e. $\frac{38982}{73}$) is merely the arithmetical form of $\frac{38x^3+9x^2+8x+2}{7x+3}$ and the Arithmetical chart has merely shown the above given algebraical operation in its arithmetical shape (wherein x=10) and that, whenever the algebraical working has taken a remainder-digit

$$
\begin{array}{l|l}
3 & :38\ \ 9\ \ 8:\ 2 \\
7 & ::\quad\ 3\ \ 3:\ 1 \\
\hline
 & :\ 5\ \ 3\ \ 4:\ 0 \\
\hline
\end{array}
$$

over to the right with a zero added, the arithmetical chart shows that particular remainder prefixed to the dight already there.

Thus, where $3x^3$ has been counted as $30x^2$ and added to the $9x^S$ already there and produced $39x^2$ as the result, this algebraical operation has been graphically pictured as the prefixing of **3** to **9** and making it **39**! And similarly, in the next step of the division, the remainder **3** is prefixed to the 8 already there; and we have to deal with **38**; and similarly, at least, the **1 prefixed** to the **2** gives us **12** (which the **3×4** subtracted therefrom cancels out)!

In other words, the given expression $38x^3+9x^2+8x+2$ (with **10** substituted for x) is actually the same as $35x^3+36x^8+37x+12$. And we say:

$$\frac{38x^3+9x^2+8x+2}{7x+3}=\frac{35x^3+36x^2+37x+12}{7x+3}=5x^2+3x+4$$ And,

graphically, this algebraical operation is demonstrated arithmetically in the manner shown in the margin.

3 :	38	9	8	:	2
7 :		3	3	:	1
:	5	3	4	:	0

The procedure is very simple and needs no further exposition and explanation. A few more illustrative instances (with running comments, as usual) will however, be found useful and helpful and are therefore given below:

(1) Divide **529** by 23.
The procedure is exactly the same and is simple and easy.

3 :	6		2	:	9
2 :		1		:	0
:	2		3	:	0

(2) Divide **4098** by **64.**

4 :	40	9	:	6	:
A t		4	:	1	:
:	6	4	:	0	:

(3) Divide **16384** by **128**

8 :	16	3	8	:	4
12:		4	11	:	6
:	1	2	8	:	0

(4) Divide 7632 by 94

(ii) ***New Nikhilam*** method or (ii) Newest Vedic method

94	: 7 6	: 3 2	:	
$\overline{06}$	: 0	: 42	:	
		: 0 3 6 :		
	: 76	: 4 8 8 :		
	: 81	: 18	:	

9^4	: 7 6 ^{3}	: ^{2} :	
	: 4	: 2 :	∴ Q=81
	: 8 1	: 18 :	R=18

(5) Divide 601325 by 76.

6	: 60 1 3 2	: 5	
7	: 11 6 2	: 2	
	: <u>7</u> 9 1 2	: 13	

Here, in the first division by 7, we can put 8 down as the first quotient-digit ; but the remainder then left will be too small for the subtraction expected at the next step. So, we take 7 as the quotient-digit and prefix the remainder 11 to the next dividend-digit. [N.B. For purposes of reference and verification, it will be a good plan to ***underline*** such a quotient-digit (because the chart offers itself for verification at every step and any reconsideration necessary at any stage need not involve our going back to the beginning and starting the whole thing over again)]

(6) Divide 3100 by 25.

5	: 3 1 0	: 0	:
2	: 1 2	: 2	:
	: 1 2 4	: 0	:

Note—In algebraic terminology, $3100 = 3x^3 + x^2 = 2x^3 + 9x^2 + 8x + 20$ and the above example is the arithmetical way of stating that $2x^3 + 9x^2y$ $8xy^2 + 20y^3 = (2x+5y)(x^2+2xy+4y^2)$ (i.e. 25 X 124=3100)

(7) Similar is the case with regard to the division of $38x^3+9x^2+8x+2$ by $(x-1)$, wherein $Q=38x^2+47x+55$ and $R=57$.

(8) Divide 695432 by 57.

7	: 6 9 5 4 3	: 2	:
5	: 1 2 1 0	: 3	:
	: 1 2 2 0 0	: 32	:

(9) Divide 3279421 by 53.

3	: 32 7 9 4 2	: 1	:
6	: 2 4 6 5	: 6	:
	: 6 1 8 7 5	: 46	:

(10) Divide 7777777 by 38

8 : 7 7 7 7 7 7 : 7 :
3 : 1 1 5 7 8 : 7 :
: 2 0 4 6 7 8 : 13 :

(11) Divide 500001 by 89.

9 : 50 0 0 0 : 1
8 : 10 [illegible] 8 : 15
: 5 6 1 7 : 88

(12) Divide 37941 by 47.

7 : 37 9 4 : 1 :
4 : 5 3 : 6 :
: 8 0 7 : 12 :

(13) Divide 745623 by 79

9 : 74 5 6 2 : 3 :
7 : 11 6 9 : 9 :
: 9 4 3 8 : 21 :

(14) Divide 7453 by 79 (to 3 places of decimals)

9 : 74 5 3 ·0 0 :
7 : 116 6 5 0 :
: 9 4 ·3 4 2 0 :

(15) Divide 710 014 by 39 (to 3 places of decimals)

9 : 7 1 0 . 0 1 4 :
3 : 4 8 2 2 6 4 :
: I 8 . 2 0 5 5 :

(16) Divide 220 by 52 (to 3 places of decimals)

2 : 2 2 0 ·0 0 0
5 : 2 2 1 4
: 4 . 2 3 0 8 :

(17) Divide 7·3 by 53 (to 5 places of decimals)

3 : 7 . 3 0 0 0 0 :
5 : 2 5 6 4 4 :
:0·1 3 7 7 3 6 :

(18) Divide 71 by 83 (to 5 places of decimals)

3 : 7 1 0 0 0 0 :
8 : 7 6 5 3 .
: 0 .85 6 4 2 ... :

(19) Divide 1337 by 79

(i) By the *New Nikhilam* method

79 : 13 : 3 7 :
$\overline{2}1$: 2 : 1
: 1 0 5 :
: 15 : 1 : 52 :
: 21 :
: 16 : : 73 :

(ii) By the newest Vedic method

9 : 13 3 : 7 :
7 : 6 : 12 :
: 1 6 : 73 :

(20) Divide **1681** by **41.**

1	:	16	8	1	:
4	:	0	0	0	:
	:	4	1	0	:

N.B.:—The Algebraical form is:

$$\frac{16x^2+8x+1}{4x+1}=4x+1$$

(21) Divide **115491** by **137.**

7	:	115	4	9	:	1	:
13	:		11	6	:	2	:
	:	8	4	3	:	0	:

or in Algebraical form:

$$13x+7 \;:\; 115x^3+\;4x^2+9x+1(8x^2+4x-3$$
$$:\; 104x^3-56x^2$$
$$11x^3-52x^2+9x=58x^2+\;9x$$
$$52x^2+28x$$
$$6x^2-19x+1$$

$$\therefore\; E=104x^3+108x^2+67x+21$$

$$\therefore\; Q=8x^2+4x+3 \text{ (i.e. 843) \& } R=0$$

$$=41x+\;1$$
$$39x+21$$
$$2x-20=0$$

(22) Divide **7458** by **127** (to 3 places of decimals)

7	:	74	5	8	0	0	0	0	0	
1 2	:		14	1	4	8	7	8	4	
	:	5	8	·7	2	4	4	.....		

(23) Divide **3517** by **127** (to 3 places of decimals)

7	:	35	1	7	0	0	0	
12	:		11	13	16	10	13	
	:	2	7	·6	9	2	......	

(24) Divide **7031985** by **823**

23	:	70	3	1	9	:	8	5		
8	:		6	7	5	:	4	4	3	and
	:	8 6	4	4			3 3			R=273

Here, the Divisor is of 3 digits. All the difference which this makes to us is that, instead of putting one extra digit on top, we put both the extra digits (23) there; and, we adopt a slightly different *modus operandi* (on the *ŪRDHVA-TIRYAK* lines) in respect of the subtraction- portion of the work.

In this instance, for instance, we divide **70** by 8 and set 8 and 6 down in their proper places as usual. Thus, our second Gross Dividend is now **63.** From that, we subtract 16 (the product of the first of the flag-digits i.e. *2* and the first quotient

digit i.e. 8) and get the remainder **(63—16=47)** as the actual dividend. And, dividing it by **8,** we have **5** and **7** as Q and R respectively and put them down at their proper places. So now, our Gross Dividend is **71**; and we deduct, by the ***Ūrdhva-Tiryak*** rule, the cross-products of the two flag-digits (23) and the two quotient-digits (85) i.e. **10+24=34**; and our remainder is **71—34=37.** We then continue to divide again by 8 and subtract etc., in the same manner (by cross-multiplication) as (just now explained) by the ***Ūrdhva-Tiryak*** method (until the last digit of the dividend) is reached. And that finishes the task.

And, in other divisions too, irrespective of the number of digits in the divisor, we follow the same method. And, in every case, our actual divisor is ***of one digit only*** (or at the most, a small two-digit one (like 12, **16** and so on) which one can easily divide by)! And all the rest of the digits (of the divisor) are hoisted on the flag-top. And this is the whole secret of the "Straight Division" formula.

Note:—If instead of the decimal places in the Quotient, you want the remainder, you can have it in the usual way. In this case, **23** and **44** (by cross-multiplication) give us **20,** which (when taken to the right) means **200**; and **3×4** (the last flag-digit × the last obtained quotient-digit)=.**12.** Subtracting the total of these two (i.e. **212**) from **485,** we have **R=273.** (i.e. **R=485—200—12=273).**

Some more instances (of division by three-digit divisions etc.) are cited below:

(1) Divide **1064321** by **743** (to 4 places of decimals)

```
43 : 10 6 4 3 2 1 0 0 :
7  :     3 4 4 5 7 7 6 5 :
   ----------------------
   : 1 4 3 2 . 4 6 4 3 :
   ----------------------
```

∴ **R=521—170—6=345**

(2) **Divide 222220** by **735** (to 3 places of decimals)

```
35 : 22  2 2 2 0 0 :
7  :      1 3 3 5 3 3 :
   --------------------
   : 3 0 2·3 4 0 :
   --------------------
```

R=320—60—10=250

(3) Divide 888 by 672 (to 3 places of decimals)

72 : 8 8 8 0 0 :
6 : 2 3 3 4 2 :
: 1 .3 2 1 5 :

or $73\bar{2}$: 8 : 8 . 8 :
: : 1 :
: 1 : R=216 :

or by mere *Vilokanam* (Inspection)

(4) 28 : 6 3 8 1 8 : 2 7 :
5 : 1 1 6 5 : :
: 1 2 0 8 6 : R=419 :

or 32 : 6 3 8 1 8 : 2 7 :
5 : 1 0 4 5 : 4 12 :
: 1 2 0 8 6 : R=419 :

(5) Divide 13579 by 975

75 : 13 5 7 9 :
9 : 4 : 11 :
: 1 3 :
R=1179−260−15=904

(6) Divide 513579 by 939

39 : 51 3 5 : 7 9 :
9 . 6 12 : 14 :
: 5 4 6 :
R=1479−540−54=594

(7) Divide 7143 by 1171

(i) By the new *Parāvartya* method

1171 : 7 : 1 4 3 :
−1−7−1 : : −7 −49 −7 :
: 7 : $\bar{6}$ 45 $\bar{4}$:
: 7 : −1054 :
: 6 : 117

(ii) By the new *Parāvartya* (Vinculum) method :

1171 : 7 : 1 4 3 :
−1−7−1 : : −14 −21 −7 :
−2+3−1 : 7 : 13 +25 −4 :
: 7 : −1054 :
: 6 : 117

(iii) *By* the newest Vedic method

71 : 71 : 4 3 : ∴ Q=6
11 : : 5 1 : and R=543−543−426=*117*
: 6 : 1 0 :

(8) Divide 4213 by 1234 (to 4 places of Decimals)

34 : 42 : 1 3 0 0 9
12 : : 6 4 7 3 2
: 3 : 4 1 4 .1 0

(9) Divide 46781 by 1483 (to 3 places of decimals)

83 : 46 : 7 8 1 0
14 : : 4 9 11 12 3
: 3 : 1 .5 4 5

(10) Divide 3124 by 1532 (to 3 places of decimals)

32 : 31 : 2 4 6
15 : : 1 6 0
: 2 . : 0 4 0

(11) Divide .333333 by 1782 (to 3 places of decimals)

82 : 33 : 3 3 3 3
17 : : 16 19 8
: 1 : 8 7

(12) Divide 46315 by 1054 (to 3 places ot decimals).

54 : 46 : 3 1 5 0
10 : : 6 13 10 8
: 4 : 3·9 4 2
R=1315—310—12

(13) Divide 75313579 by 1213

13 : 75 : 3 1 3 5 79
12 : : 3 3 11 11 11
: 6 : 2 0 8 8
R=1179—344=835

(14) Divide 135791 by 1245

46 : 13 : 5 7 91
12 : : 1 11 4
: 1 : 0 9

R=491—405 or 130—44=86

(15) Divide 13579 by 1616

16 : 135 : 79
16 : : 7
: 8 :

R=779—128 or 690—39=651

(16) Divide 135791 by 1632

32 : 135 7 : 91
16 : 7 : 5
: 8 3 :

R=591—250—6=335

(17) Divide 97531 by 1627

27 : 97 5 : 31
16 : 17 : 21
: 5 9 :
R=2131—593
=1538

or

27 : 97 5 : 31
16 : 1 : 3
: 6 0 :
and R=331—420=—89
i.e. Q=59 and R=1538

(18) Divide 97531 by 1818

18: 97 5 : 31
18 . 7 : 16
: 5 3 :

∴ R=1631—454 (or 1200—23)=1177

(19) Divide 13579 by 2145

45 : 135 : 79
21 : : 9
: 6 :

And R=979—270 or 930—221=709

(20) Divide 135791 by 2525

25 : 135 7 : 91
25 : 10 : 22
: 5 3 :

And R=2291—325 or 1980—14
=1966

(21) Divide 50 X 11 by 439 (to three places of decimals).

39	: 50 ·11 0 :		$\overline{4}\overline{1}$	: 50 : ·110 :	R=19$\overline{8}$=182
4	: 1 33 6 :	or	4	: 1 : ·222 :	or 311—129
	: 11 ·415 :			: 11 : ·415 :	And=182

(22) Divide 15×61 by 349 (to three places of Decimals).

49	: 15X 6 1 0		5$\overline{1}$	: 15 .6 1 0 :
3	: 3 8 85 10	or	3	: 3 4 4 :
	: 4 . 475 277			: 4 47277 :

And R=361—196
or 200—35=165

And R=361—196=165

(23) Divide 47 by 798 (to five places of decimals).

98	: ·4 7 0 0 0 0		0$\overline{2}$	:·4 7 0 0 0 0
7	: 4= 121915	or	8	: 4 7 6
	: · 0 5 8 9 0			:·0 5 8 9 0 ··

(24) Divide 1111 by 839

39	: 11 : 1 1 :	or
8	: : 3 4 :	
	: 1 : 3 :	By mere *Vilokanam* (Inspection)

And R=311—39=272

We now extend the jurisdiction of the *Sūtra* and apply it to Divisors consisting of a large number of digits. The principle involved being the same, the procedure is also identically the same as in the foregoing examples. And the division by a single digit (or a small two-digit divisor) continues exactly the same. A few illustrative instances are given hereunder :

(1) Divide 7031.95 by 8231 (to 5 decimal places).

231	: 70 3 1 9 5 0 0
8	: 6 7 . 5 4 6 9
	: ·8 5 4 32

(2) Divide 995 311 by 16123

123	: 99 5 : 3 1 1
16	: 3 : 13
	: 6 1 :

R=13311—1503 (or 1200+$\overline{190}$ +$\overline{2}$)=11808

(3) Divide 975 311 by 16231

```
231 : 97  5 : 3  1  1
16  :     1 : 3
    ----------------
    :  6  0 :
    ================
```

R $=1500+\overline{50}+1$ or $3311-1860=1451$

(4) Divide 975 311 by 16 333

```
333 : 97   5  :  3  1  1
16  :     17  : 16
    --------------------
    :  5   9  :
    ====================
```

R $=16311-4647=11664$
or $12100+\overline{410}+\overline{26}=11664$

(5) Divide 975 311 by 18123

```
123 : 97   5 :  311
18  :      7 : 16
    ----------------
    :  5   3 :
    ================
```

R $=15000+\overline{200}+\overline{8}$
(or $16311-1519$) $=14792$

(6) Divide 995311 by 20321

```
321 : 99   5 :  311
20  :     19 : 2 3
    ----------------
    :  4   8 :
    ----------------
```

R $=20100-190-7$ ($=23311-3408$) $=19903$

(7) Divide 997531 by 30321

```
321 : 99   7   531
30  :      9   28
    ----------------
    :  3   2
    ----------------
```

R $=27300-40-1$
(or $28531-1272$) $=27259$

(8) Divide 137 294 by 5749 (to 6 places of decimals).

```
749 : 13  7  2   9   4    0   0   0
5   :     3  8  13  13   14  14   8
    --------------------------------
    :  2  3·  8   8   1  370
    --------------------------------
```

or

```
351 : 13  7·  2   9   4   0   0   0
6   :     1       5   3   2   5   3   0
    ------------------------------------
    : 23  ·   88      1370
    ------------------------------------
```

(9) Divide 53 247 by 4999 (to five places of dectmals).

```
999 : 53.  2   4   7   0   0
4   :  1   4   9   11  1412
    ---------------------------
    : 10.  6   5   1   5   3 ......
    ---------------------------
```

or

```
001-: 5   3 . 2   4   7   0   0   0
6   :     0   3   2   0   2   1   0
    ---------------------------------
    : 1   0.6   5   1   5   3 ...
    ---------------------------------
```

N.B. Better to divide by **50**

(10) Divide 13 8462 by 39898 (to 3 places of Decimals).

```
9898 : 13   8   4   6   2   0   0
3    :      4 9 1 3 1 4     0
     ----------------------------
     :  3.  4   7   0   0 3
     ----------------------------
```

or

```
0102 : 13.  8   4   6   2 :
4    :      1   2   3   4 :
     ----------------------
     :  3.  4   7   03  9 :
     ----------------------
```

N.B. Better to divide by **40**

(11) Divide 131 by 19799 (to **5** places of decimals).

```
9799:   1   3   1   0   0   0
1   :       1   7   11  13  19
    ---------------------------
    : 0 0   6   6   1   6 ... ...
    ---------------------------
```

or

```
0201 :  1  3  1 0 0 0
a    :     1  1
     -----------------
     : 0 0  6  6 16
     -----------------
```

N.B. Better to divide by 20

(12) Divide 76432 by 67998 (to **5** places of decimals).

```
7998 : 7   6   4   3   2   0
6    :     1   3   6   7   9   1   0
     -------------------------------
     : 1 1   2   4   0   3
     -------------------------------
```

or

```
2002 : 7   6   4   3   2   0   0
7    :     0   1   2   6   7   18
     -------------------------------
     : 1.  1   2   4   0   3 ......
     -------------------------------
```

(13) Divide ·2537 by 48329

8329: 2 5 3 7 0 or 2331: ·2 5 3 7 0 0
4 : 5 5 1 0 1 6 5 : 2 0 3 1 1

: .0 5 2 4 9 : .0 6 2 4 9 ...

(14) Divide 371628.112 by 12734 (to 6 decimal places).

2784 : 371628· 112
1 : 14 8 7 11 8 12 8 11 11 11

: 291839 25868

N.B. Better divide by 12

or

$\bar{3}34$: 3 7 1 6 2 8· 112 00
13 : 1 1 0 1 4 3 1 4 5 8 6

: 2 9 1 8 3 9 2 6 8 6 8

N.B. **Here** we have divided by 13

(15) Divide 4'1326 by 31046 (to **5** decimal places).

1046 : 4· 1 3 2 6 0 0 or 3 1054 : 4· 1 3 2 6 0 0
3 : 1 1 1 2 4 : 1 1 1 1 1 1

:1· 3 3 1 1 2 ... :1· 3 3 1 1 2...

(16) Divide ·20014 by 137608 (to **5** decimal places).

37608 : ·2 0 0 1 4 0 or 42412 : 2 0 0 1 4 0
1 : 1 3 6 8 9 1 : 1 2 1 0

: · 1 4 5 44 ... : · 1 4 5 44 ...

N.B. Better divide ***by*** 13

(17) Divide **.0034147** by 814256321 (to 6 decimal places).

14266321 : 0 0 3 4 1 4 7
8 3 2 9 5

: ·0 0 0 0 419 ......

N.B. The Vinculum method is **always** available but will not make much difference. In fact, it may prove stiffer.

(18) Divide ·**200103761** by 9371836211 (to 3 places).

371838211 : ·2 0 0 1 0 3 7 6 1
9 : 2 6 7 7

: ·0 2 1 3 6 ... ...

(19) Divide 74. 5129 by 9314

```
314 : 74·  5  1  2  9  0  0  0   0
9   :      2  1  3  0  9  9  9  9
    ------------------------------
    : 8·      0  0  0  0  0  966
    ------------------------------
```

(20) Divide 71324 by 23145

```
3145 : 7 :  1  3  2  4    or    145 : 71 :  3  2  4
2    :   : 1  2  4  4           23  :    : 2  20  6 18
     ------------------               ------------------
     : 3 :  0  8  1 ...               : 3 :  0  8  1
     ------------------               ------------------
```

(21) Divide 137426 by 743·2×1·242 X 80·04 (to 4 places of decimals).

```
432 : 13   7 : 4   2   6   0   0   0
7   :  6     : 7  11   7   5   4   8
    ---------------------------------
    : 184    :  9   1   1   1   9
    ---------------------------------
42  :           6  12  15  15   7  11
    ---------------------------------
12  : 14            8   8   8  17
    ---------------------------------
004 :           6   4   0   4   9
    ---------------------------------
8   .  1·       8 6    0   0   9
    ---------------------------------
    : =1·       8 6    0   1          (approximately)
    ---------------------------------
```

CHAPTER XXVIII

AUXILIARY FRACTIONS

In our exposition of vulgar fractions and decimal fractions, we have so far been making use of processes which help to give us the *exact* results in each case. And. in so doing, we have hitherto (generally) followed the current system whereby multiplications and divisions by powers of ten are mechanically effected by the simple device of putting the decimal point backwards or forwards (as the case may be).

Conventional Method

For instance, we manipulate the decimal point thus:

(1) $\frac{1}{800}=\frac{\cdot 01}{8}$ (2) $\frac{39}{70}=\frac{3\cdot 9}{7}$; (3) $\frac{17}{130}=\frac{1\cdot 7}{13}$;

(4) $\frac{3741}{110000}=\frac{\cdot 3741}{11}$; and (6) $\frac{97654}{90000000}=\frac{\cdot 0097654}{9}$

But after this has been done, the other operations—of division etc.,—have had to be carried out in the usual manner.

Auxiliary Fractions

There are certain Vedic processes, however, by which, with the aid of what we call SAHAYAKS (AUXILIARY fractions), the burden of the subsequent operations is also considerably lightened and the work is splendidly facilitated.

First Type

The first (and commonest) type thereof is a **very** simple and easy application of our self-same old friend the ***Ekādhika Pūrva***. And the whole ***modus operandi*** is to replace the Denominator by its ***Ekādhika*** (i.e. to drop the last digit and increase the penultimate one by 1) and make a consequential alteration in the division-procedure (as in the case of other ***Ekādhika*** operations).

N.B. :—The student will remember that, in these operations, the remainder at each step (of a division) 's not prefixed to a series of zeroes from the right-hand side, but to each quotient-digit.

Auxiliary Fractions (*A.F.*) (*First type*)

(1) for $\frac{1}{19}$, the Auxiliary fraction is $\frac{1}{2}$

$\therefore \frac{1}{19} = \cdot\dot{0}52631578/94736842\dot{1}$

(2) for $\frac{1}{29}$, the A.F. is $\frac{1}{3}$

$\therefore \frac{1}{29} = \frac{1}{3(AF)} = \cdot\dot{0}3448275862068/965517241379\dot{3}\dot{1}$

(3) for $\frac{37}{59}$ AF is $3\cdot7/6$

(4) for $\frac{3}{59}$, AF is $\frac{\cdot 03}{6}$

(5) for $\frac{73}{89}$, AF is $\frac{7\cdot 3}{9}$

(6) for $\frac{1}{119}$, AF is $\frac{\cdot 1}{12}$

(7) for $\frac{1}{149}$, AF is $\frac{\cdot 1}{15}$

(8) for $\frac{7}{149}$, AF $= \frac{\cdot 07}{15}$

(9) for $\frac{172}{1299}$, AF $= \frac{1\cdot 72}{13}$

(10) for $\frac{371}{7999}$, AF is $\frac{\cdot 371}{8}$

(11) for $\frac{537}{89999}$, AF is $\frac{\cdot 0537}{9}$

(12) for $\frac{56}{15999}$, AF is $\frac{\cdot 056}{16}$

(13) tor $\frac{50}{69999}$, AF is $\frac{\cdot 0050}{7}$

(14) for $\frac{1}{14999999}$, AF $= \frac{\cdot 000001}{15}$

(15) for $\frac{2175}{79999999}$, AF $= \frac{\cdot 0002175}{8}$

(16) for $\frac{21863}{49999}$, AF $= \frac{2\cdot 1863}{5}$

In the above' oases, the first eight denominators end in a single nine ; the remaining eight terminate in 2, 3, 4, 3, 4, 6, 7 and 4 nines respectively. The queation now is : Does it stand to reason that the *Ekādhika* should be the same in $\frac{7}{29}$ and in $\frac{7}{2999}$ (irrespective of the difference in the number of nines) ? That would be tantamount to declaring that the same (significant) numerator (or dividend) with two different denominators (or divisors) will yield the same quotient ! And that would be palpably absurd !

Yes ; the objection is perfectly valid ; and the relevant Sūtra has surmounted this difficulty beforehand, by providing for groups of quotient-digits (to which the remainder at each stage of the mental division should be prefixed)! And that solves the whole problem.

Modus Operandi

For instance, let us take the sixteenth example supra (namely, $\frac{21863}{49999}$ whose A.P. is $\frac{2\cdot1863}{5}$ and whose denominator ends in four nines):

Here, F is $\frac{21863}{49999}$; and AF is $\frac{2\cdot1863}{5}$; and we have to make 5 (in lieu of 49999) our working divisor. As we have dropped 4 nines from the original denominator and have 5 as our *Ekādhika* in the Denominator of the Auxiliary fraction, we have to divide the numerator of the latter in bundles, so to say, of 4 digits each by 5; and, whatever remainder there is, has to be prefixed not to any particular quotient-digit but to the bundle just already reached.

Thus, we take up 2·1863 to start with and divide it by 6. We get ·4372 as the first quotient-group and 3 as the remainder. We prefix this remainder to that group and say:

$$\left[\begin{array}{l} 5)\ \underline{2'1863} \\ Q = \cdot4372 \quad -R=3 \end{array}\right.$$

$_3\cdot4372$ and we divide this dividend (namely, 34372) by the same divisor 5; and we get:

$$_4.4372 \ : \ _2 6874 \ \big| \ _4 5374$$

i.e. 6874 is the second quotient-group; and 2 is the second remainder, which therefore we prefix to the second quotient group. And we continue this process with as many groups as we need.

Thus we have: $_3\cdot4372,\ _2 6874,\ _4 5374,\ _4 9074$ and so on (to any number, or tens, or hundreds or thousands etc., of Decimal places)!

The PROOF hereof is very simple:

$$\begin{array}{rcl} 000\bar{1} & : & 2\cdot1863\ 000 \\ 5 & : & 3 \\ \hline & : & \cdot\ 4372\ \ldots \\ \hline \end{array}$$

$\therefore$ F $= \cdot4372,\ 6874,\ 5374,\ 9074\ldots\ldots$

N.B. :—The prefixed remainders are not parts of the quotient but only prefixes to the quotient-group in question and are therefore to he dropped out of the answer!

This is a simple method by which we avoid divisions by long big divisors and have small and easy denominators to deal with.

The student will note that division by big denominators (with a continuous series of *zeroes* on the right-hand side) and division by the *Ekādhika* (with the *prefixing* of the remainder at each step) yield the same result ! And this is why the **Auxiliary** Fraction scheme has been incorporated for lightening the burden of long big divisions.

A few more examples are given below:

(1) Express $\frac{6}{29}$ in its decimal shape.

Here, $F=\frac{6}{29}$ $\therefore$ A.F.$=\frac{\cdot 6}{3}$

$\therefore$ F$=\cdot$20689655172413 79310344827586

(2) $F=\frac{71}{89}$ $\therefore$ $AF=\frac{7\cdot1}{9}$ $\therefore$ F$=\cdot$79775280898 etc., etc.

(3) $F=\frac{17}{139}$ $\therefore$ $AF=\frac{1\cdot7}{14}$ $\therefore$ F$=\cdot$1 2 2 3 0 215827......
3 3 4 0 3

(4) $F=\frac{98}{179}$ $\therefore$ $AF=\frac{9\cdot8}{18}$ $\therefore$ F$=\cdot$5474860335, 1955......

(5) $F=\frac{1}{43}=\frac{3}{129}$ $\therefore$ $AF=\frac{\cdot3}{13}$ $\therefore$ F$=\cdot$023255813953488...

(6) F=17/43=51/129 $\therefore$ $AF=\frac{5\cdot1}{13}$ $\therefore$ F$=\cdot$395 348 837 209...

(7) $F=\frac{18}{73}=\frac{54}{219}$ $\therefore$ $AF=\frac{5\cdot4}{22}$ $\therefore$ F$=\cdot$246 57534......

(8) $F=\frac{53}{799}$ $\therefore$ $AF=\frac{\cdot53}{8}$ $\therefore$ F$=\cdot$06 63 32 91 61 45 18...
5 2 7 4 3 1

N.B. :—The upper row ('06633291614518......) is the answer and the lower one (5 2 7 4 3 1......) is a mere scaffolding and goes out.

(9) $F=\frac{15}{899}$ $\therefore AF=\frac{\cdot 15}{9}$ $\therefore$ F= 01 66 85 20 57
6 7 1 5 7

(10) $F=\frac{2}{1799}$ $\therefore AF=\frac{\cdot 02}{18}$ $\therefore$ F= 00 11 11 72 87...
2 2 13 15 6

(11) $F=\frac{100}{233}=\frac{300}{699}$ $\therefore AF=\frac{3}{7}$ $\therefore$ F= 42 91 84 54 93 56 22
6 5 3 6 3 1 2
31.........
5

(12) $F=\frac{444}{13999}$ $\therefore AF=\frac{\cdot 444}{14}$ $\therefore$ F= 031 716 551 182...
10 1 2 3

(13) $F=\frac{97017}{29999999}$ $\therefore AF=\frac{\cdot 0097017}{3}$

F= 00323391 00107791 6670259 etc., etc
0 2 2

The student will have noted that the denominators in all the above cases ended in 9 (or 3 which could be so multiplied as to yield an easy multiple ending in 9). But what about those ending in 1 which would have to be multiplied by 3 for this purpose and would, therefore, as already pointed out (in the chapter on Recurring Decimals) yield a rather unmanageable *Ekādhika*? is there any provision for this kind of fractions?

Yes; there is. And this takes us on to the second type of Auxiliary fractions.

Auxiliary Fractions (Second Type)

If and when F has a denominator ending in 1, drop the 1 and DECREASE the numerator by unity. This is the required second type of Auxiliary Fractions. Thus,

(1) for 3/61, AF=2/60= $\frac{2}{6}$
(2) for 36/61, AF=35/60=3 5/6
(3) for 28/71, AF=27/70=2·7/7
(4) for 73/91, AF=72/90=7·2/9
(5) for 2/121, AF=1/120=·1/12
(6) for 14/131, AF=13/130=1·3/13
(7) for 1/301, AF=0/300=·00/3
(8) for 1/901, AF=0/900=·00/9
(9) for 17211301, AF=171/1300=1·71/13

(10) for 2743/7001, AF=2742/7000=2·742/7
(11) for 6163/8001, AF=6162/8000=6·162/8
(12) for 1768/9001, AF=1767/9000=1·767/9
(13) for 56/16001, AF=55/16000=·055/16
(14) for 50/700001, AF=49/700000=·00049/7
(15) for 2175/80000001, AF=2174/80000000=·0002174/8
(16) for 1/900000001, AF=0/900000000=·00000000/9

Modus operandi

The principles, the prefixing (to the individual quotient-digits or to groups of quotient-digits) etc., and other details are the same as in the *Ekādhika* Auxiliary fraction. BUT the ***procedure*** is different, in a very important (nay, vital) particular. And this is that after the first division (or group-division is over) we prefix the remainder not to each quotient-digit but to its COMPLEMENT from NINE and carry on the division in this way all through.

An illustrative instance will clarify this:

Let F be $\frac{13}{31}$ $\therefore$ AF$=\frac{12}{30}=\frac{1\cdot 2}{3}$

(i) We divide 1.2 by **3** and set **4** down as the first quotient-digit and **0** as the first remainder.
·**4**
0

(ii) We then divide not 04 but 05 (the complement of 4 from 9) by **3** and put **1** and **2** as the second quotient-digit and the second remainder respectively. Therefore we have
·**4 1**
0 2

(iii) We take now, not **21** hut 28 as our dividend, divide it by 3 and get:
·**4 1** 9
0 2 1

(iv) Thus, dividing **10** by 3, we have:
9
4 1 9 3
0· 2 1 1 and so on, until finally our chart reads:

F (i.e. $\frac{13}{31}$)= ·4 1 9 3 5 4 8 3 7 **0** 9 6 etc., etc. !
0 2 1 1 1 2 1 2 2 0 2 2 2

Always, therefore, remember to take the complement (from 9) of each quotient-digit (and not the quotient-digit itself) for the purpose of further division, subtraction etc. This is the whole secret of the second type of Auxiliary Fraction.

Some more illustrative examples are given hereunder :

(1) $F=\frac{1}{41}$ $\therefore$ $AF=\frac{\cdot 0}{4}$

$\therefore$ F = $\cdot\dot{0}$ 2 4 3 $\dot{9}$/0
0 1 1 3 0/0

} *So,* this is a definite recurring decimal.

(2) $F=\frac{70}{71}$ $\therefore$ $AF=\frac{6.9}{7}$

$\therefore$ F = ·9 8 5 9 1 5 4 9 2 9 5 7 7 4 6 4 7 8 8 7 3 2 3...
6 4 6 1 3 3 6 2 6 4 5 5 3 4 3 5 6 6 5 2 1 2 6

(3) $F=\frac{91}{171}$ $\therefore$ $AF=\frac{9\cdot 0}{17}$ $\therefore$ F = ·5 3 2 1 6 3 7 4 2 6
5 3 2 10 6 12 7 4 11 15

(4) $F=\frac{10}{27}=\frac{30}{81}$ ... $AF=\frac{2\ 9}{8}$ $\therefore$ $\cdot\dot{3}$ 7 $\dot{0}$/3
5 0 2/5

Evidently a recurring decimal.

(5) $F=\frac{131}{701}$ $\therefore$ $AF=\frac{1\cdot 30}{7}$ (with groups of 2 digits)

$\therefore$ F = ·18 68 75 89 15 83......
4 5 6 1 5 3

(6) $F=\frac{1400}{1401}$ $\therefore$ $AF=\frac{13.99}{14}$ (with two-digit groups)

F $\therefore$ = ·99 92 86 22 41
13 12 3 5 3

(7) $F=\frac{243}{1601}$ $\therefore$ $AF=\frac{2.42}{16}$ (with groups of two digits)

$\therefore$ F = ·15 17 80 13 74 14 11 61 79......
2 12 2 11 2 1 9 12 4

(8) $F=\frac{5}{67}=\frac{15}{201}$ ... $AF=\frac{\cdot 14}{2}$ (with groups of two digits)

$\therefore$ F = ·07 46 26 86 56
0 0 1 1 1

(9) $F=\frac{2743}{7001}$ $\therefore$ $AF=\frac{2\cdot742}{7}$ (with 3-digit groups)

$\therefore$ F= ·391 801 171 261 248 393 086
            5   1   1   1   2   0   4

Proof: 001 : 2 742 999
            7 :   5
              ——————
              : 5391
              ——————

(10) $F=\frac{31}{77}=\frac{93}{231}$ $\therefore$ $AF=\frac{9\cdot2}{23}$ $\therefore$ F= $\dot{4}$ 0 2/5 9 $\dot{7}$
                                                                  0

OR $F=\frac{31}{77}=\frac{403}{1001}$ $\therefore$ $\frac{\cdot402}{1}$ (with three-digit groups)

$\therefore$ F= ·$\dot{4}$02/59$\dot{7}$ (evidently a recurring decimal)

(11) $F=\frac{29}{15001}$ $\therefore$ $AF=\frac{\cdot0128}{15}$ (with three-digit groups)

$\therefore$ F= ·001 933 204 453 036 etc., etc.
          1   3   3   6   0

(12) $F=\frac{137}{13000001}$ $\therefore$ $AF=\frac{\cdot000136}{13}$ (with 6-digit groups)

$\therefore$ F= ·000010 : 538460 : 727810 : 713245 etc., etc.
        6   : 9   : 0   : 4

Other Astounding Applications

Yes; but what about still other numbers which are neither immediately below nor immediately above a ten-power base or a multiple of ten etc., (as in the above cases) but a bit remoter therefrom? Well; these too have been grandly catered for, in the shape of a simple application of the *Ānurūpya* Sūtra, whereby, after the pre-fixing of each' Remainder to the quotient-digit in question, we have to add to (or subtract from) the dividend at every step, as many times the quotient-digit as the divisor (i.e. the denominator) is below (or above) the NORMAL which, in the case of all these Auxiliary fractions, is counted as ending, not in zero or a number of zeroes but in 9 or a series of nines!

For example, let F he $\frac{15}{68}$ and suppose we have to express this vulgar fraction in its decimal shape (to, say, 16 places of decimals).

Lest the student should have, in the course ot these peregrinations into such very simple and easy methods of work, forgotten the tremendous difference between the current method and the Vedic method and thereby deprived himself of the requisite material for the purpose of comparison and contrast, let us, for a brief while, picture the two methods to ourselves side by side and see what the exact position is.

According to the Vedic method, the process wholly *mental* is as follows:

$F=\frac{15}{68}$, $\therefore$ A.F. $=\frac{1\cdot5}{7}$. But 68 being one less than 69 (the normal ending in 9) we shall have to add to each dividend, the quotient-digit in question. Thus

(i) when we divide 1·5 by 7, we get 2 and 1 as our first quotient-digit and our first remainder.

1.5
7
·2
1

(ii) our second dividend will not be 12 hut 12+2=14; and by division of that by 7, our second Q and R are 2 and 0.

·2 2
1 0

(iii) our next dividend is 02f 2=04; and this gives us 0 and 4 as Q and R.

·2 2 0
1 0 4

(iv) our fourth dividend is 40+0, giving us 6 and 5 as our fourth Q and R.

·2 2 0 5
1 0 4 5

(v) So, our next dividend is 55+5(=60); and our Q and R arc 8 and 4.

·2 2 0 5 8
1 0 4 5 4

We can proceed on these lines to as marly places of decimals as we may need. And, in the present case (wherein 16 decimal-places have been asked for), we toss offdigit after digit (mentally) and say:

F= ·2 2 0 5 8 8 2 3 5 2 9 4 1 1 7 6 etc., etc.
1 0 4 5 4 0 2 3 1 6 1 0 1 5 3 2

Over against this, let us remind ourselves of the current method for answering this question :

```
68)15·0(·2205882352941176 etc., etc.
   136
   ---
    140
    136
    ---
     400
     340
     ---
      600
      544
      ---
       560
       544
       ---
        160
        136
        ---
         240
         204
         ---
          360
          340
          ---
           200
           136
           ---
            640
            612
            ---
             280
             272
             ---
               80
               68
               --
              120
               68
               520
               476
               ---
                440
                ---
                408
                ---
```

Alongside of this cumbrous 16-step process, let us once again put down the *whole working* by the Vedic method and say :

$F\left(\frac{15}{68}\right)=$ ·2 2 0 5 8 8 2 3 5 2 9 4 1 1 7 6 4 etc., etc.
1 0 4 5 4 0 2 3 1 6 1 0 1 5 3 2 4

A few more illustrative examples are given hereunder:

(1) Express 101/138 in its decimal shape (20 places).

(i) *Routine Method*—

```
138)101· 0(·73188405797101449275
     966
     ---
      440
      414
       260
       138
       ---
       1220
       1104
       ----
        1160
        1104
        ----
          560
          552
          ---
            800
            690
            ---
            1100
             960
             ----
             1340
             1232
             ----
               980
               966
               ---
                140
                138
                ---
                  200
                  138
                  ---
                   620
                   552
                   ---
                    680
                    552
                    ---
                    1280
                    1242
                    ----
                      380
                      276
                      ---
                      1040
                       966
                        740
                        690
                        ---
                         50
                         --
```

(ii) ***Vedic Method.***

A.F. $=\frac{10.1}{14}$ (with one below the normal 139)

∴ F = ·7 3 1 8 8 4 0 5 7 9 7 1 0 1 4 4 9 2 7 4 etc.
3 2 1 2 1 0 4 0 8 1 0 1 2 8 0 0 2 6 6 1 2 2 1 0 6 5

Note:—More-than-one-digit quotient if any, should be carried over (as usual) to the left.

(2) ***Conventional method*** :

97)73·0(·75257731958762886597 etc., etc.

```
679
---
 510                    740
 485                    679
 ---                    ---
  250                    610
  194                    582
  ---                    ---
   560                    280
   485                    194
   ---                    ---
    750                    860
    679                    776
    ---                    ---
     710                    840
     679                    776
     ---                    ---
      310                    640
      291                    582
      ---                    ---
       190                    580
        97                    485
       ---                    ---
        930                    950
        873                    873
        ---                    ---
         570                    770
         485                    679
         ---                    ---
          850                    91
          776
          ---
```

(ii) ***Vedic*** (at-sight) method

Data : F $=\frac{73}{97}$ ∴ AF $=\frac{7.3}{10}$ (but with 2 below the normal 99).

∴ Add twice the Q-digit at each step.

Actual Working :

∴ F = ·7 5 2 5 7 7 3 1 9 5 8 7 6 2 8 8 6 5 9 7 etc., etc.
1 5 6 5 1 1 9 3 7 5 4 1 8 6 4 4 8 5 7

(3) Express $\frac{17}{127}$ as a decimal (20 places).

Current *method*—

127)17·0(·13385826771653543306 **etc.,** eto,

127
430
381
490
381
1090
1016
740
635
1050
1016
340
254
860
762
980
889
910
889
210
127
830
762
680
635
450
381
690
635
550
508
420
381
390
381
900
882
18

(ii) ***Vedic*** at-sight method—

∴ $F=\frac{17}{127}$ ∴ $AF=\frac{1\cdot7}{13}$ (but with 2 below tne normal 129)

∴ Double the Q-digit to be added at every step.

∴F=·1 3 3 8 5 8 2 6 7 7 1 6 5 3 5 4 3 3 0 6etc.
4 4 1 0 5 9 1 8 8 7 0 8 5 3 6 4 3 3 0 9 1 2

(4) Express $\frac{5236}{8997}$ in decimal form (21 decimal places)

(i) ***Usual method—***

8997)5236·0(·581/971/768/367/233/522/285 etc., etc.

44985	
73750	
71976	
17740	
8997	
87430	21010
80973	17994
64570	30160
62979	26991
15910	31690
8997	26991
69130	46990
62979	44985
61510	20050
53982	17994
75280	20560
71976	17994
33040	25660
26991	17994
60490	76660
53982	71976
65080	46840
62979	44985
	1855

What a TREMENDOUS mass and mess of multiplications, subtractions etc. !

(ii) ***Vedic at-sight method*:**

$\because F = \frac{5236}{8997}$ $\therefore AF = \frac{5 \cdot 236}{9}$ (but with 2 below the normal 8999 and also with groups of 3 digits at a time).

$\therefore$ Add twice the Q-digit at every step.

$\therefore$ F = ·581	: 971	: 768	: 367	: 233	: 522	: 285 etc., etc.
7	: 4	: 1	: 1	: 4	: 1	: 1

(5) Express $\frac{21863}{49997}$ as a decimal (16 places)

Conventional method

```
49997)21863·0(·4372/8623/7174/2302/etc., etc.
      199988
      ------
       186420
       149991
       ------
        364290
        349979
        ------
         143110
          99994
         ------
          431160
          399976
          ------
           311840
           299982
           ------
            118580
             99994
            ------
             185860
             149991
             ------
              358690
              349979
              ------
                87110
                49997
               ------
                371130
                349979
                ------
                 211510
                 199998
                 ------
                  115120
                   99994
                  ------
                   151260
                   149991
                   ------
                    126900
                     99994
                    ------
                     26906
                    ------
```

What a horrible mess ?

(ii) *Vedic At-sight method*—

$\because$ F$=\frac{21863}{49997}$ $\therefore$ AF$=\frac{2\cdot1863}{5}$

(with 2 below the normal 49999 and with groups of four digits each).

$\therefore$ Add double the **Q-digit** at every step)

$$\therefore F = 4372 : \overline{8\ 5(12)}^{62}\ 3 : \overline{6(11)}^{71}\ \overline{73}^{4} : \overline{(12)}^{2}\ \overline{2\ (10)\ 2}^{30}$$
$$3 \quad :1 \quad :4 \quad :4$$

N.B. : very carefully that the extra (or surplus i.e. left-hand side) parts of Q-digits have been "carried over" to the left.

This **excess** is due to the additional multiplication and can be got over in the manner just indicated. **A** method for *avoiding* this difficulty altogether is also available but will be dealt with at a later stage.

(6) Express $\frac{17}{76}$ as a decimal (eight places)

Current method—

76)17·0(·22368421 etc., etc.
152
180
152
280
228
520
456
640
608
320
304
160
152
80
76
4

N.B.—Note 84 : 21 :: 4 : 1.

Even this is bad enough.

(ii) Vedic ***At-sight method*** :

$\because$ $F = \frac{17}{76}$ $\therefore$ $AF = \frac{1 \cdot 7}{8}$ (but with 3 less than the normal 79)

Thrice the Q-digit is to be added at every step.

$\therefore$ F = ·**2** 2 3 6 8 4 2 **1** etc., etc.
1 2 4 4 0 0 0 0

(7) **Express** $\frac{17125}{59998}$ as a decimal (**12** places)

(i) ***Usual method***—

```
59998)17125·0(·2854/2618/0872
       119996
       ------
        512540
        479984
        ------
         325560
         299990
         ------
          255700
          239992
          ------
           157080
           119996
           ------
            370840
            359988
            ------
             108520
              59998
             ------
              485220
              479984
              ------
               523600
               479984
               ------
                436160
                419986
                ------
                 161740
                 119996
                 ------
                  41744
                 ------
```

(ii) *Vedic At-sight method* :.

$\because F = \frac{17125}{59998}$ $\therefore AF = \frac{1 \cdot 7125}{6}$ (with 1 less the normal and with 4-digit groups)

∴ only one Q-digit is to be added.

∴ F= ·2854 : 2618 : 0872 etc., etc.
 1 : 0 : 4

These examples should suffice to bring vividly home to the student the extent and magnitude of the difference between the current cumbrous methods and the Vedic at-sight one-line process in question.

Yes ; but what about other numbers, in general, which are nowhere near any power or multiple of ten or a "normal" denominator-divisor ending in 9 or a series of nines ? Have they been provided for, too ?

Yes ; they have. There are methods whereby, as explained in an earlier chapter (the one dealing with recurring decimals) we can easily transform any miscellaneous or non-descript denominator in question—by simple multiplication etc.,—to the requisite standard form which will bring them within the jurisdiction of the Auxiliary Fractions hereinabove explained.

In fact, the very discovery of these Auxiliaries and of their wonderful utility in the transmogrification of frightful -looking denominators of vulgar fractions into such simple and easy denominator-divisors must suffice to prepare the scientifically-minded seeker after Knowledge, for the marvellous devices still further on in the offing.

We shall advert to this subject again and expound it still further, in the next two subsequent chapters (dealing with DIVISIEILITY and the application of the *Ekādhika Pūrva* etc., as positive and negative OSCULATORS in tliat context).

CHAPTER XXIX

DIVISIBILITY AND SIMPLE OSCULATORS

We now take up the interesting (and intriguing) question as to how one can determine before-hand whether a certain given number (however long it may be) is divisible by a certain given divisor and especially as to the Vedic processes which can help us herein.

The current system deals wth this subject but only in an ultra-superficial way and only in relation to what may be termed the most elementary elements thereof. Into details of these (including divisibility by **2, 5, 10, 3, 6, 9, 18, 11, 22** and so on), we need not now enter (as they are well-known even to the mathematics-pupils at a very early stage of their mathematical study.) We shall take these for granted and start aith the intermediate parts and then go on to the advanced portions of the subject.

The Osculators

As we have to utilise the "वेष्टनs" (*Veṣṭanas* =Osculators) throughout this subject (of divisibility), we shall begin aith a simple definition thereof and the method of their application.

Owing to the fact that our familiar old friend the *Ekādhika* is the first of these osculators (i.e. the positive osculator), the task becomes all the simpler and easier. Over and above the huge number of purposes which the *Ekādhika* has already been shown to fulfil, it has the further merit of helping us to readily determine the divisibility (or otherwise) of a certain given dividend by a certain given divisor.

Let us, for instance, start with our similar familiar old friend or experimental-subject (or shall we say, "Guinea-pigs" the number **7**. The student need hardly be reminded that the *Ekādhika* for **7** is derived from $7\times7=49$ and is therefore *6*. The *Ekādhika* is a clinching test for divisibility ; and the process by which it serves this purpose is technically called *Veṣṭana* or "Osculation".

Suppose we do not know and have to determine whether 21 is divisible by 7. We multiply the last digit (i.e. 1) by the *Ekādhika* (or Positive Osculator i.e. 5) and add the product (i.e. 5) to the previous digit (i.e. 2) and thus get 7. This process is technically called *"Osculation"*. And, if the result of the osculation is the divisor itself (or a repetition of a previous result), we say that the given original dividend (21) is divisible by 7.

A trial chart (for 7) will read as follows:

14; $4\times5+1=21$; and $1\times5+2=7$ ∴ YES.
21 (already dealt with);
28; $8\times5+2=42$; $2\times5+4=14$ (already dealt with)
35; $5\times5+3=28$ (already dealt with);
42 (already dealt with);
49; $9\times5+4=49$. (*Repetition means divisibility*).
56; $6\times5+5=35$ (already dealt with);
63; $3\times5+6=21$ (already dealt with);
70; $0\times5+7=7$ ∴ YES
77; $7\times5+7=42$ (already done);
84; $4\times5+8=28$ (already over);
91; $1\times5+9=14$ (already dealt with);
98; $8\times5+9=49$ (already done);
Now let us try and test, say, 112.
112; $2\times5+1=11$; $11\times5+1=56$ ∴ YES.
OR $2\times5+11=21$ ∴ YES.

We next try and test for 13; and we find the repetitions more prominent there. The *Ekddhika* is 4. Therefore we go on multiplying leftward by 4. Thus,

13; $3\times4+1=13$
26; $6\times4+2=26$
39; $9\times4+3=39$
52; $2\times4+5=13$
65; $5\times4+6=26$
78; $8\times4+7=39$
91; $1\times4+9=13$
104; $4\times4+10=26$

The repetition etc., is uniformly there and in correct sequence too (i.e. 13, 26, 39)! ∴ YES.

Examples of the Osculation Procedure (Veṣṭana)

A few examples will elucidate the process:

(1) 7 continually osculated by 5 gives 35, 28, 42, 14, 21 and 7.

(2) 5 so osculated by 7 gives 35, 38, 59, 68, 62; 20, 2 and so on.

(3) 9 (by 7) gives 63, 27, 51, 12, 15 etc.

(4) 8 (by 16) gives 128, 140, 14 etc.

(5) 15 (by 14) gives 71, 21, 16 etc.

(6) 18 (by 12) gives 97, 93, 45, 64, 64, 53, 41, 16 etc.

(7) 36 (by 9) gives 57, 68, 78, 79, 88, 80, 8 etc.

(8) 46 (by 3) gives 22, 8 etc., etc.

(9) 49 (by 16) gives 148, 142, 46, 100, 10, 1 etc., eta.

(10) 237 (by 8) gives 79, 79 etc., and is .. divisible by 79.

(11) 719 (by 9) gives 162, 33, 30, 3 etc., etc.

(12) 4321 (by 7) gives 439, 106, 52, 19, 64, 34, 31, 10, 1 etc.

(13) 7524 (by 8) gives 784, 110, 11, 9 etc., etc.

(14) 10161 (by 5) gives 1021, 107, 45, 29, 47, 39, 48, 44, 24, 22, 12, 11, 6 etc.

(15) 35712 (by 4) gives 3579, 393, 51, 9 etc.

(16) 50720 (by 12) gives 5072, 531, 65, 66, 78, 103, 46, etc., etc.

N.B. :—We need not carry this process indefinitely on. We can stop as soon as we reach a comparatively small number which gives us the necessary clue as to whether the given number is divisible (or not) by the divisor whose *Ekādhika* we have used as our osculator I Hence the importance of the *Ekādhika*.

Rule for Ekādhikas

(1) For 9, 19, 29, 39 etc., (all ending in 9), the *Ekādhikas* are 1, 2, 3, 4 etc.

(2) For 3, 13, 23, 33 etc., (all ending in 3) multiply them by 3; and you get 1, 4, 7, 10 etc., ae the *Ekādhikas*.

(3) For 7, 17, 27, 37 etc., (all ending in 7) multiply them by 7; and you obtain 5, 12, 19. 26, etc., *as* the *Ekādhikas*.

(4) For 1, 11, 21, 31, etc., (all ending in 1), multiply them by **9**; and you get 1, **10, 19, 28** etc., as the *Ekādhikas.*

Osculation by own *Ekādhika*

Note that the osculation of any number by its own *Ekādhika* will (as in the case of 7 and 13) go on giving that very number or a multiple thereof. Thus,

(1) 23 osculated by 7 (its *Ekādhika*) gives $7\times3+2=23$;
46 (osculated by 7) gives $7\times6+4=46$;
69 (similarly) gives $7\times9+6=69$;
92 (likewise) gives $2\times7+9=23$;
115 (similarly) gives $7\times5+11=46$ And so on.

Now, 276 (osculated by 7) (by way of testing for divisibility by 23) gives $7\times6+27=69$ which again gives 69! ∴ YES. Thus, all the multiples of 23 fulfil this test i.e. of osculation by its *Ekādhika* (7). And this is the whole secret of the *Veṣṭana* sub-Sūtra.

Modus Operandi of Osculation

Whenever a question of divisibility comes up, we can adopt the following procedure. Suppose, for instance, we wish to know—without actual division—whether 2774 is divisible by 19 (or not). We put down the digits in order as shown below. And we know that the *Ekādhika* (osculator) is 2.

(i) We multiply the last digit (4) by 2, add the product (8) to the previous digit 7 and put the total (15) down under the second right-hand digit.

2	7	7	4
		15	

(ii) We multiply that 15 by **2**, add that 30 to the 7 on the upper row, cast out the nineteens (from that 37) and put down the remainder 18 underneath that 7.

2	7	7	4
	37	15	
	(18)		

N.B.—This casting out of the nineteens may be more easily and speedily achieved by first osculating the 15 itself, getting 11, adding it to the 7 (to the left-hand) on the top-row and putting the 18 down thereunder.

2	7	7	4
	18	15	

(iii) Wc then osculate that 18 with the 2 to the left on the upper row and get 38; or we may osculate the 18 itself, obtain 17, add the 2 and get 19 as the final. osculated result. And, as 19 is divisible by 19, we say the given number (2774) is also divisible thereby.

$\begin{pmatrix} 2 & 7 & 7 & 4 \\ 19 & 18 & 15 & \end{pmatrix}$

This is the whole process; and our chart says:

By 19 ?
∴ The osculator is 2 $\left\{\begin{matrix} 2 & 7 & 7 & 4 \\ 19 & 18 & 15 & \end{matrix}\right\}$ ∴ YES.

OR Secondly, we may arrive at the same result-as effectively but less spectaculariy by means of a continuous series of osculations of the given number (2774) by the osculator (2) as hereinbefore explained. And we can say:

∵ 2774 (osculated by the osculator **2**) gives us 285. 38 and 19

∴ 2774 is divisible by 19.

N.B. :—The latter method is the shorter but more mechanical and cumbrous of the two; and the former procedure looks neater and more pictorially graphic, nay, spectacular. And one can follow one's own choice as to which pocedure should be preferred.

Note:—Whenever, at any stage, a bigger number than the divisor comes up, the same osculation-operation can always be performed.

Some more specimen examples are given below:

(1) By 29 ? ∴ The osculator is 3. $\left\{\begin{matrix} 3 & 2 & 8 & 9 & 6 \\ 27 & 8 & 31 & 27 & \end{matrix}\right\}$ ∴ No.

ORE (osculated by 3) gives 3307 351, 38, 27, etc. ∴ No.

(2) By 29 ? ∴ The osculator is 3. $\left\{\begin{matrix} 9 & 3 & 1 & 4 & 8 \\ 29 & 26 & 27 & 28 & \end{matrix}\right\}$ ∴ YES

OR The osculation-results are 9338, 957, 116 and 29
∴ YES

(3) By 29 ? ∴ The osculator is 3. $\begin{Bmatrix} 2 & 43 & 45 & 21 \\ 29 & 9\ 21 & 6\ 20 & 5 \end{Bmatrix}$ ∴ YES

OR The osculation-results are 243455, 24360, 2436 261 and 29 ∴ YES

(4) By 39 ? ∴ The osculator is 4. $\begin{Bmatrix} 4 & 9 & 1 & 4 \\ 39 & 38 & 17 & \end{Bmatrix}$ ∴ YES

OR The osculation-results are 507, 78 and 39 ∴ YES

(5) By 49 ? ∴ The osculator is 5. $\begin{Bmatrix} 5 & 3 & 3 & 2 \\ 51 & 19 & 13 & \\ (10) & & & \end{Bmatrix}$ ∴ No.

OR The osculation-results are 543, 69, 51 and 10 ∴ No

(6) By 59 ? ∴ The osculator is 6. $\begin{Bmatrix} 1 & 9 & 1 & 5 & 7 & 3 \\ 59 & 49 & 46 & 37 & 25 & \end{Bmatrix}$ ∴

OR The osculation-results are 19175, 1947, 236 and 59 ∴ YES

(7) By 59 ? ∴ The osculator is 6. $\begin{Bmatrix} 1 & 2 & 5 & 6 & 7 \\ 59 & 49 & 57 & 48 & \end{Bmatrix}$ ∴ YES

OR The osculation-results are 1298, 177 and 59 ∴ YES.

(8) By 59 ? ∴ The osculator is 6. $\begin{Bmatrix} 4 & 0 & 1 & 7 & 9 & 1 \\ 47 & 17 & 52 & 38 & 15 & \end{Bmatrix}$ ∴ No.

OR Theosculation-results are 40185,4048,452 and 57 ∴No.

(9) By 79 ? ∴ The osculator is 8. $\begin{Bmatrix} 6 & 3 & 0 & 9 & 4 & 8 & 2 & 1 \\ 13 & 70 & 38 & 64 & 76 & 9 & 10 & \end{Bmatrix}$ ∴ No.

OR The osculation-results are 6309490, 630949, 63166, 6364, 668, 130 and 13 ∴ No.

(10) By 43 ? ∴ The osculator is 13. $\begin{Bmatrix} 1 & 4 & 0 & 6 & 1 \\ 129 & 119 & 118 & 19 & \end{Bmatrix}$ ∴ YES

OR The osculation-results are 1419,258 and 129 ∴ YES

(11) By 53 ? ∴ The osculator is 16. $\begin{Bmatrix} 2 & 1 & 9 & 5 & 3 \\ 149 & 39 & 62 & 53 & \end{Bmatrix}$ ∴ No.

OR The osculation-results ar 2243, 272 and 59 ∴ No.

(12) By 179 ? ∴ Theosculatoris 18. $\begin{Bmatrix} 7 & 1 & 45 & 5 & 0 & 1 \\ 179 & 109 & 6 & 20 & 150 & 18 \end{Bmatrix}$ ∴ YES

OR The osculation-results are 714668, 71600, 7160, 716 and 179 ∴ YES

(13) Determine whether 5293240096 is divisible by 139 (or not)

(A) By the current method (just by way of contrast) :

```
139)5293240096(380864
    417
    ----
    1123
    1112
    ----
      1124
      1112
      ----
        1200
        1112
        ----
         889
         834
         ---
          556
          556
          ---
            0       ∴ YES.
```

(B) By the *Vedic method* :

By 139 ? ∴ The *Ekādhika* (osculator) is 14.

5	2	9	3	2	4	0	0	9	6	YES
139	89	36	131	29	131	19	51	93		

OR The osculation-results are 529324093, 62932451 5293259 529451,52959, 5421, 556 and 139 ∴ YES

Note:—In all the above cases, the divisor either actually ended in 9 or could—by suitable multiplication—be made to yield a product ending in 9 (for the determination of the required *Ekādhika* or Osculator in each case). But what about the numbers ending in 3,7 and 1 (whose *Ekādhika* may generally be expected, to be a bigger number) ? Is there a suitable provision for such numbers being dealt with (without involving bigger Ekidhika-multipliers) ?

Yes ; there is ; and this we proceed to deal with.

The Negative *Osculator*

This is an application of the *Parāvartya* Sūtra and is called the Negative Osculator because it is a process not of *addition*

(as in the case of of the *Ekādhika*) but of Subtraction (leftward). And this actually means a consequent alternation. of *plus* and minus.

Examples of *the Negative Osculation Process*

(1) **36** thus osculated by 9, gives **3—54=—51.**

(2) 7(osculated by 5) gives **0—35=—35**

(3) 35712 (osculated by 4) will yield **8—3571=—3563.**

How to Determine *the* Negative *Osculator*

Just as the *Ekādhika* (the positive *Veṣṭana*) has been duly defined and can be correctly ascertained, similarly the Negative Osculator will also require to be determined by means of a proper definition and has been so defined with a view to proper recognition.

It consists of two clauses:

(i) In the case of all divisors ending in 1, simply drop the one; and

(ii) in the other cases, multiply so as to get 1 as the last digit of the product (i.e. 3 by 7, 7 by 3 and 9 by 9); and then apply the previous sub-clause (i.e. drop the 1).

Note:—For facility of symbolisation, the positive and the negative osculators will be represented by P and Q respectively.

Examples of Negative Osculators

(1) For 11, 21, 31, 41, 51 and other numbers ending in 1, Q is 1, 2, 3, 4, 5 and so on. [Note that, by this second type of oscultors, we avoid the big *Ekādhikas* (produced by multiplying these numbers by 9)].

(2) For 7, 17, 27, 37, 47, 57 etc., we have to multiply them by 3 (in order to get products ending in 1). And they will be 2, 5, 8, 11, 14, 17 and so on. (In these cases too, this process is generally calculated to yield smaller multipliers than the multiplication by 7 is likely to do).

(3) For 3, 13, 23, 33, 43, 63 etc., we havetomultiply them by 7; and the resultant Negative Osculators will be 2, 9, 16, **23,** 30, **37** etc., (which will generally be found to be bigger numbers than the ***Ekādhikas***).

(4) For 9, 19, 29, 39, **49, 59** etc., we have to multiply these by **9**; and the resultant Negative Osculators will be **8,** 17, 26, **35, 44, 53** etc., (all of which will be much bigger than the corresponding Ekadhikas).

Important ***and*** Interesting Feature

Note:—A very beautiful, interesting and important feature about the relationship between F and Q, is that, whatever the Divisor (D) may be, P+Q=D. i.e. the two osculators together invariably add up to the Divisor. And this means that, if one of them is known, the other is automatically known (being the complement thereof from the divisor i.e. the Denominator).

Specimen Schedule of Osculators *P* and Q

Number	Multiple for P	Multiple for Q	P	Q	Total
1	9	(1)	1	0	1
3	9	21	1	2	3
7	49	21	5	2	7
9	(9)	81	1	8	9
11	99	(11)	10	1	11
13	39	91	4	9	13
17	119	51	12	5	17
19	(19)	171	2	17	19
21	189	(21)	19	2	21
23	69	161	7	16	23
27	189	81	19	8	27
29	(29)	261	3	26	29
31	279	(31)	28	3	31
33	99	231	10	23	33
37	259	111	26	11	37
39	(39)	351	4	35	39
41	369	(41)	37	4	41
43	129	301	13	30	43
47	329	141	33	14	47
49	(49)	441	5	44	49
51	459	(51)	46	5	51

Number	Multiple for P	Multiple for Q	P	Q	Total
53	159	371	16	37	53
57	399	171	40	17	57
59	(59)	531	6	53	59
61	549	(61)	55	6	61
63	189	441	19	44	63
67	469	201	47	20	67
69	(69)	621	7	62	69
71	639	(71)	64	7	71
73	219	511	22	51	73
77	539	231	54	23	77
79	(79)	711	8	71	79
81	729	(81)	73	8	81

N.B. It will be noted :—

(i) that P+Q always equals D;

(ii) multiples of 2 and 5 are inadmissible for the purposes of this schedule;

(iii) and these will have to be dealt with by dividing off all the powers of 2 and 5 (which are factors of the Divisor concerned).

A** few **sample examples

(1) for 59, P is 6 ∴ Q= 53
(2) for 47, Q is 14 ∴ P= 33
(3) for 53, P is 16 ∴ Q= 37
(4) for 71, Q s 7 ∴ P= 64
(5) for 89, P is 9 ∴ Q= 80
(6) for 83, P is 25 ∴ &= 58
(7) for 91, P is 82 ∴ Q= 9
(8) for 93, Q is 65 ∴ P= 28
(9) for 97, P is 68 ∴ Q= 29
(10) for 99, Q is 89 ∴ P= 10
(11) for 101, P is 91 ∴ Q= 10
(12) for 103, Q is 72 ∴ P= 31
(13) for 107, P is 75 ∴ Q= 32
(14) for 131, Q is 13 ∴ P=118
(15) for 151, P is 136 ∴ Q= 15
(16) for 201, Q is 20 ∴ P=181

P+Q=D throughout

Note :—(1) If the last digit of a divisor be 3, its P < its Q ;
(2) If the last digit be 7, its Q <its P ; and
(3) in the actual working out of the subtractions of the osculated multiples (for the negative osculators), the actual result will be an alternation of ***pks*** and ***minus.***

Explanation (1) In the removal of brackets, a series of subtractions actually **materialises** in an alternation of + and— For example,

$$a-(b-\{c-[d-\overline{e-f})\}]$$
$$=a-b+c-d+e.$$

Exactly similar is the case here.

(2) When we divide a^n+b^n by $(a+b)$, the quotient consists of a series of terms which are alternately **plus** and minus. Exactly the same is the case here.

Note:—The student will have to carefully ***remember*** this alternation of **positives** and negatives. But the better thing will be, not to rely on one's memory at each step but to mark the digits beforehand, alternately, say, **by** means of a Vinculum (from right to left), on all the even-place digits, so that there may be an automatic safeguard against the possible playing of any **pranks** by one's memory.

Armed with this safeguard, let us now tackle a few illustrative instances and see how the plan works out in actual practice.

(1) By 41 ? ∴ Tho (Negative) **Osculator** is 4

		3	
—	4 1		∴ YE8

Or

The osculation-results are 1$\underline{6}$564, 1640,164 and 0 ∴ YES

(2) By 31 ? ∴ Q—3 $\left\{\begin{matrix}\bar{6} & 6 & 0 & 3\\ 0 & 33 & 9 & \end{matrix}\right\}$:.YES

Or

The osculation-results are 651, 62 and 0 ∴ **YES.**

(3) By 41 ? ∴ Q=4 $\left\{\begin{matrix} 1 & \bar{1} & 2 & \bar{3} & 4 \\ 0 & 10 & 13 & 13 & \end{matrix}\right\}$ ∴ YES

Or

The osculation-results are 1107, 82 and 0 ∴ YES

(4) By 47 ? ∴ Q=14 $\left.\begin{matrix} \bar{7} & 4 & \bar{2} & 1 & \bar{6} & 5 \\ 111 & 102 & 7 & 51 & 64 & \end{matrix}\right\}$ ∴ SO

Or

The osculation-results are 74146, 7330, 733 and 31 ∴ NO

(5) By 51 ? ∴ Q=5 $\left\{\begin{matrix} \bar{4} & 3 & \bar{7} & 3 & \bar{2} & 1 \\ -5 & 10 & 32 & 18 & 3 & \end{matrix}\right\}$. NO

Or

The osculation-results are 43727, 4337, 398 & I ∴ NO

(6) By 61 ? ∴ Q=6 $\left\{\begin{matrix} 1 & \bar{9} & 5 & \bar{8} & 1 \\ 0 & -51 & -7 & -2 & \end{matrix}\right\}$ ∴ YES

Or

The osculation-results are 1952, 183 and 0 ∴ YES

(7) By 67 ? ∴ Q=20 $\left\{\begin{matrix} \bar{1} & 0 & \bar{1} & 7 & \bar{1} & 2 & \bar{0} & 3 \\ 0 & -10 & 1 & 0 & -81 & -4 & 60 & \end{matrix}\right\}$ ∴ YES

Or

The osculation-results are 1017060, 101706, 10050, 1005 and 0 ∴ YES.

(8) By 91 ? ∴ Q=9 $\left\{\begin{matrix} 9 & \bar{8} & 0 & \bar{4} & 5 & \bar{9} & 0 & \bar{5} & 3 \\ 84 & 69 & 49 & 56 & 37 & 44 & 16 & 22 & \end{matrix}\right\}$ ∴ NO

Or

The osculation-results are 98045878, 9804515, 980406 97986, 97441 938 and 21 ∴ NO

(9) By 61 ? ∴ Q=6 $\left\{\begin{matrix} 1 & 2 & 2 & 1 & 3 & 0 & \bar{5} & 4 \\ 0 & -\overline{10} & -\bar{2} & 0 & 1053 & 19 & & \end{matrix}\right\}$ ∴ YES

Or

The osculation-results are 1221281, 122122, 12200, 1220, 122 and 0 ∴ YES

(10) By 71 ? ∴ Q=7 $\left\{\begin{matrix} 8 & 0 & \bar{9} & 0 & \bar{4} & 5 \\ 0 & 62 & 19 & 4 & 31 & \end{matrix}\right\}$ ∴ YES

Or

The osculation-results are 80869, 8023, 781 and 71 ∴ YES

(11) By 131 ? ∴ Q=13 $\left\{\begin{matrix} 1 & \bar{3} & 3 & \bar{7} & 9 & \bar{0} & 3 \\ 0 & 10 & 1 & 20 & 123 & 39 & \end{matrix}\right\}$ ∴ YES

Or

The osculation-results are 133751, 13362, 1310 & 131 ∴ YES

(12) By 141 ? ∴ Q=14 $\left\{\begin{matrix} 4 & \bar{8} & 9 & \bar{8} & 8 & \bar{5} & 7 \\ 94 & 87 & 37 & 2 & 41 & 93 & \end{matrix}\right\}$ ∴ NO

N.B. But this dividend (yielding the same results) is divisible by 47 (whose Q is also 14). (94=47×2)

CHAPTER XXX

DIVISIBILITY

AND

COMPLEX MULTIPLEX OSCULATORS

The cases so far dealt with are of a simple type, involving only small divisors and consequently small osculators. What then about those wherein bigger numbers being the divisors, the osculators are bound to be correspondingly larger ?

The student-inquirer's requirements in this direction form the subject-matter of this chapter. It meets the reeds in question by formulating a scheme of *groups* of digits which can be osculated, not as individual digits but in a lump, so to say.

Examples of *Multiplex Veṣṭana i.e.* (Osculation)

(1) 371 osculated by 4 for 2 digits at a time, gives 3+ 71×4 (=287) and 3—284 (=—281) for plus oscillation and minus oscillation respectively.

(2) 1572 osculated by 8 for 2 digits gives 15+576 (=591) and 15—576 (=—561) respectively.

(3) 8132 osculated by 8 (P and Q) for 2 digits gives 81 +256 (=337) and 81—256 (=—175) respectively.

(4) 75621 osculated by 5 (P and Q) for 3 digits gives 75+3105 (=3180) and 75—3105 (=—3030) respectively.

(5) 61845 osculated by 7 (P and Q) for 3-digit groups gives 61+5915 (=5976) and 61—5915 (=—5854) respectively.

(6) 615740 osculated by 8 (P and Q) for 3-digit packets gives 615+5920 (=6535) and 615—5920 (= —5305) respectively.

(7) 518 osculated by 8 (P and Q) for 4-digit bundles gives 0+4144 (=4144) and 0—4144 (——4144) respectively.

(8) 73 osculated by 8 (P and Q) for five-digit groups yields 0+584 (=584) and 0—584 (=—584) respectively.

(9) 210074 osculated by 8 (P and Q) for five-digit bundles give **2+80592 (=80594)** and 2—80592 (= —80590) respectively.

(10) 7531 osculated by **2(P)** for 3 digitsgives **7+1062=1069**

(11) 90145 osculated by 5 (Q) for 3 gives **—725+90=—635**

(12) 5014112 osculated by 7 **(Q)** for 4 gives **501—28784** =—28283

(13) 7008942 osculated by 3 **(P)** for 2 gives **126+70089 =70215**

(14) 7348515 osculated by 8 **(P)** for 3 gives **7348+4120 =11468**

(15) 59076242 osculated by 7 **(Q)** for **2** gives **—590762+294** = —590468

Categories of Divisors and their Osculators.

In this context, it should be noted that, as there are various types of divisors, there are consequent differences as to the nature and type of osculators (positive and/or negative) which will suit them. They are generally of two categories:

(i) those which end in nine (or a series of nines) [in whch case they come within the jurisdiction of the. ***Ekādhika*** (i.e. the Positive) Osculator] or, which terminate in or contain series of zeroes ending in **1**, (in which case they come within the scope of operations performable with the aid of the *Viparīta* (i.e. the negative) osculator ; and

(ii) those which, by suitable multiplication, yield a multiple of either of the two sorts described in sub-section (i) supra and can thus be tackled on that basis.

The First Type.

We shall deal, **first,** with the first type of divisors, namely, those ending in **9** (or a series of nines) or **1** (or a series of zeroes ending in unity) and explain a technical terminology and symbology which will facilitate our operations in this context.

(1) Let the divisor be 499. It is obvious that its osculator P is 5 and covers 2 digits. This fact can be easily expressed in symbolical language by saying : $P_2=5$

(2) In the case of 1399, it is obvious that our osculator (positive) is 14 and covers 2 digits ∴ $P_2=14$.

(3) As for 1501, Q obviously comes into play, is 15 and covers 2 digits. In other words, $Q_2=15$.

(4) For 2999, P is 3 and covers 3 digits ∴ $P_3=3$

(5) For 5001, $Q_3=5$

(6) For 7001, $Q_3=7$

(7) For 79999, $P_4=8$

(8) For 119999, $P_4=12$

(9) For 800001, $Q_5=8$

(10) For 900001, $Q_5=9$

(11) For 799999, $P_5=8$

(12) For 120000001, $Q_7=12$

Correctness of the Symbology

The osculation-process invariably gives us the original number itself (or a multiple thereof) or zero : For example,

(i) 499 (with $P_2=5$) gives us 4$5 (99)=4+495=499

(ii) 1399 (with $P_2=14$) gives 13+14 (99)=13+1386=1399

(iii) 1501 (with $Q_2=15$) gives 15×1−15=0

(iv) 2999 with $P_3=3$) gives 2+3 (999)=2+2997=2999

(v) 5001 (with $Q_3=5$) gives 5×1−5=0

(vi) 7001 (with $Q_3=7$) gives 7×1−7=0

(vii) 79999 (with $P_4=8$) gives 7+8 (9999)=79999

(viii) 119999 (with $P_4=12$) gives 11+12 (9999)=119999

(ix) 800001 (with $Q_5=8$) gives 8×1−8=0

(x) 900001 (with $Q_5=9$) gives 9×1−9=0

(xi) 799999 (with $P_5=8$) gives 7+8 (99999)=799999

(xii) 120000001 (with $Q_7=12$) gives 12×1−12=0

N.B. :—The osculation-rule is strictly adhered to ; and the P's and the Q's invariably yield the original dividend itself and zero respectively !

Utility and Significance of the Symbology

The symbology has its deep significance and high practical utility in our determining of the divisibility (or otherwise) of a certain given number (however big) by a certain given divisor (however large), inasmuch as it throws light on (1) the number of digits to be taken in each group and (2) the actual osculator itself in each individual case before us.

A few simple examples of each sort will clarify this:

(1) Suppose the question is, Is 106656874269 divisible by 499 ?

Here, at sight, $P_2=5$. This means that we have to split the given expression into 2-digit groups and osculate by 5. Thus,

$$\left\{\begin{array}{cccccc} 10 & 66 & 56 & 87 & 42 & 69 \\ 499 & 497 & 186 & 525 & 387 & \end{array}\right\} \therefore \text{ YE8}$$

($69\times5=345$; $345+42=387$; $435+3+87=525$; $5\times25+5+56=186$; $5\times86+66=497$; $5\times97+4+10=499$!)

Or

The osculation-results are 1066569087, 10666125, 106786, 1497 and 499 $\therefore$ YES

(2) Is 126143622932 divisible by 401 ?

Here $Q_2=4$

$$\therefore \left\{\begin{array}{cccccc} \overline{12} & 61 & \overline{43} & 62 & \overline{29} & 32 \\ -16 & 400 & 185 & 458 & 99 & \end{array}\right\} \therefore \text{ NO}$$

Or

The osculation-results are 1261436101, 12614357, 125915, 1199 and -385. $\therefore$ **NO**

(3) Is 69492392 divisible by 199 ?

Here $P_2=2$

$$\left\{\begin{array}{ccc} 69 & 49 & 92 \\ 199 & 463 & \end{array}\right\} \therefore \text{ YES}$$

or

The osculation-results give 695107, 6965 and 199 $\therefore$ YES

(4) Is 1928264569 divisible by 5999 ?

Here $P_3=6$

$\therefore \left\{ \begin{array}{llll} 1 & 928 & 264 & 569 \\ 5999 & 4999 & 3678 & \end{array} \right\} \therefore$ YES

Or

The osculation-results are 1931678 and 599 ∴ YES

(5) Is 2188 6068 313597 divisible by 7001 ?

Here $Q_3=7$

$\therefore \left\{ \begin{array}{lllll} 21 & 886 & 068 & \overline{313} & 597 \\ 0 & -3 & 6127 & 3866 & \end{array} \right\} \therefore$ YES

Or

The osculation-results give 21 886 064 134, 21885126, 21003 and 0. ∴ YES.

(6) Is 30102 1300602 divisible by 99 ?

Here $P_2=1$

As $P_2=1$ and continuous multiplications by 1 can make no difference to the multiplicand, the sum of the groups will suffice for our purpose :

$\because 30+10+21+30+06+2=99$ ∴ YES.

The second method amounts to the same thing and need not be put down.

(7) Is 2130 1102 1143 4112 divisible by 999 ?

Here $P_3=1$

and $\because 2+130+110+211+434+112=999$

∴ (by both methods) YES.

(8) Is 7631 3787 858 divisible by 9999 ?

Here $P_4=1$

and $\because =763+1378+7858=9999$

∴ (By both methods), YES.

(9) Is 2037760003210041 divisible by 9999 ?

Here $P_4=1$

and $\because 2037+7600+0321+0041=9999$

∴ (By both methods) YES.

(10) Is 5246 7664 0016 201452 divisible by 1001 ?

Here $Q_3=1$; and

∵ $S=\overline{524}+676+\overline{648}+\dot{0}16-201+452=221=13\times17$

∴ Divisible by 13

But $1001=7\times11\times13$ ∴ Divisible by 13 but not by 7 or by 11. ∴NO

The Second Category

The second type is one wherein the given number is of neither of the standard types (which P and Q readily and instantaneously apply to) but requires a multiplication for the transformation of the given number to either (or both) of the standard forms and for the ascertaining of the P and Q (or both) suitable for our purpose in the particular case before us.

The Process of Transformation

In an earlier chapter (the one on "Recurring Decimals") we have shown how to convert a given decimal fraction into its vulgar-fraction shape, by so multiplying it as to bring a series of nines in the product. For example, in the case of $\cdot\dot{1}4285\dot{7}$, we had multiplied it by 7 and got $\cdot\dot{9}9999\dot{9}$ (=1) as the product and thereupon argued that, because 7× the given decimal =1, ∴ that decimal should be the vulgar fraction $\frac{1}{7}$

$$(1)\quad \cdot\dot{1}4285\dot{7}\times7$$
$$=\cdot999999=1$$
$$\therefore x=\tfrac{1}{7}$$

Similarly, with regard to $\cdot\dot{0}7692\dot{3}$, we had multiplied it by 3 (in order to get 9 as the last digit of the product) ; argued that, in order to get 9 as the penultimate digit, we should add 3 to the already existing 6 there and that this 3 could be had only by multiplying the original given decimal by 1; then found that the product was now a series of nines; and then we had argued that,, ∵ $13x=1$, ∴ x must be equal to $\frac{1}{13}$. And we had also given several more illustrations of the same kind (for demonstrating the same principle and process).

$$(2)\quad \begin{array}{r} \cdot\dot{0}7692\dot{3} \\ 13 \\ \hline \cdot23076\dot{9} \\ 0\cdot76923 \\ \hline \dot{9}9999\dot{9}=1 \end{array}$$
$$\therefore x=\tfrac{1}{13}$$

As P (in the present context) requires, for osculation, numbers ending in **9** (or a series of nines), we have to adopt a similar procedure for the same purpose; and, in the case of Q too, we have to apply a similar method for producing a number which will terminate in **1** or a series of zeroes ending in **1**.

The Modus Operandi

A few examples of both the kinds will elucidate the process and help the student to pick up his **P** and **Q**. And once this is done, the rest will automatically follow (as explained above).

(1) Suppose the divisor is **857**. ∵ **857×7=5999**, we can therefore at once say : $P_3=6$.

The test and proof of the correctness hereof is that any multiple of the divisor in question must necessarily fulfil this condition i.e. on osculation by P_3, must yield **857** (or a multiple thereof).

For instance, let us take **857×13 = 11141**. As $P_3=6$ ∴ **11+6 (141)=857** ! And this proves that our osculator is the correct one.

(2) Let us now take **43**. ∵ **43×7=301**, ∴ $Q_2=3$.

Taking **43×3 (=129)** for the test, we see **129** yields **29×3−1=86**; and **80** is a multiple of **43** (being exactly double of it). So, our Q is correct.

The significance of this fact consists in the natural consequence thereof, namely, that any number (which is really divisible by the divisor in question must obey this rule of divisibility by the P process or the Q process.

N.B.:—Remember what has already been explained as regards P or Q being greater.

In this very case (of **43**), instead of multiplying it by **7**, getting **301** as the product and ascertaining that $Q_2=3$ is the osculator, we could also have multiplied the **43** by **3**, got **129** as the product, found $P_1=13$ to be the positive osculator and verified it. Thus, in the case of **43×2=86**, ∴ **8+6(13)=86** ∴ $P_1=13$ is the correct positive osculator.

Multiplication by 13 at every step being **necessarily** more cumbrous than by 3, we should naturally prefer $Q_2=3$ (to $P_1=13$).

In fact, it rests with the student to choose between P and Q and (in view of the bigness or otherwise of multiplier-osculator etc.) decide which to prefer.

(3) Ascertain the P and the Q for 137.

137	$\therefore P_2=37$	137	$\therefore Q_3=14$
27		103	
959		411	
274		1370	
3699		14111	

Obviously $Q_3=14$ is preferable (to $P_2=37$).
(Test: $137\times8=1296$ $\because$ Q_3 gives $1296-0=1296$!)

(4) Determine the P and the Q for 157.
$\because$ $157\times7=1099$ $\therefore P_2=11$
And $157\times93=14601$ $\therefore$ $Q_2=146$.
$\therefore$ $P_2=11$ is to be preferred
(Test :—$157\times7=1099$ $\therefore$ P_2 gives $10+1089=1099$)

(5) Find out the P and Q for 229
$\because$ $229\times131=29999$ $\therefore$ $P_4=3$
This **Osculator** being so **simple,** the Q need not be tried at all. But on principle, $\because$ $229\times69=15801$
at all. But on principle, $\therefore$ $229\times69=15801$
$\therefore$ $Q_2=159$ (obviously a big multiplier)
(Test for $P_4=3$
$229\times100=2/2900$ $\therefore$ $P=3$ gives $8702=229\times38$)

(6) Find P and Q for 283
$\because$ $283\times53=14999$ $\therefore$ $P_3=15$
and $\because$ $283\times47=13301$ $\therefore$ $Q_3=133$
$\therefore$ $P_3=15$ is preferable

(Test $283\times4=1132$; $1+15$ $(132)=1981=283\times7$)

(7) Find P and Q for 359

$\because$ 359 $\times$ 61 = 21899 $\therefore$ P_2 = 219

and $\because$ 359 $\times$ 339 = 14001 $\therefore$ Q_3 = 14

Obviously Q_3 = 14 is to be preferred.

(Test: (i) 359 $\times$ 3 = 1077 $\therefore$ Q_3 = 14 gives 14 $\times$ 77 − 1 = 1077

and (ii) 359 $\times$ 115 = 41285 $\therefore$ Q_3 = 14 give 14 $\times$ 285 − 41 = 3949 = 359 $\times$ 11)

(8) Ascertain P and Q for 421

$\because$ 421 $\times$ 19 = 7999 $\therefore$ P_3 = 8

and $\because$ 421 $\times$ 81 = 34101 $\therefore$ Q_2 = 341

obviously P_3 = 8 is the better one

(Test: 421 $\times$ 5 = 2105 $\therefore$ P_3 = 8 gives 2 + 840 = 842 = 421 X2)

(9) Determine P and Q for 409

$\because$ 409 $\times$ 511 = 208999 $\therefore$ P_3 = 209

and $\because$ 409 $\times$ 489 = 200001 $\therefore$ Q_5 = 2

Obviously the Q osculator is preferable.

(Test: 409 $\times$ 1000 = 4/09000

$\therefore$ Q_5 = 2 gives 18000 − 4 = 17996 = 409 $\times$ 44)

Having thus studied the multiplex osculator technique and *modus operandi*, we now go on to and take up actual examples of divisibility (which can be easily tackled by the multiplex osculatory procedure).

Model Applications to Concrete Examples

(1) Is 79158435267 divisible by 229?

$\because$ 229 $\times$ 131 − 29999 $\therefore$ P_4 = 3

$\therefore$ $\begin{Bmatrix} 791 & 5843 & 5267 \\ 5725 & 21644 & \end{Bmatrix}$

But 5725 = 229 X 25 $\therefore$ YES

(2) Is 6056200566 divisible by 283?

$\because$ 283 $\times$ 53 = 14999 $\therefore$ P_3 = 15

$\because$ $\begin{Bmatrix} 6 & 056 & 200 & 566 \\ 6226 & 10414 & 8690 & \end{Bmatrix}$

But 6226 = 283 $\times$ 22 $\therefore$ YES

(3) Is 7392 60251 divisible by 347 ?

$\therefore$ 347×**317=109999** $\therefore$ $P_4=11$

$$\left\{\begin{array}{lll} 7 & 3926 & 6251 \\ 73654 & 0627 & \end{array}\right\}$$

But **73654=347×212** $\therefore$ YES

(4) Is 867 311 7259 divisible by 359 ?

$\because$ **359×39=14001** $\therefore$ $Q_3=14$

$$\therefore \left\{\begin{array}{llll} \overline{} & 673 & \overline{117} & 259 \\ 1 & 764 & 3509 & \end{array}\right\} \therefore \text{NO}$$

(5) Is 885648437 divisible by 367 ?

$\because$ **367×3=1101** $\therefore$ $Q_2=11$

$$\therefore \left\{\begin{array}{lllll} 8 & \overline{85} & 64 & \overline{84} & 37 \\ 734 & 66 & 314 & 323 & \end{array}\right\}$$

But **734=367** X2 $\therefore$ YES

(6) Is 490 222 8096 divisible by 433 ?

$\because$ **433×3=1299** $\therefore$ $P_2=13$

$$\therefore \left\{\begin{array}{llllll} & & 02 & 22 & 80 & 96 \\ 5 & 7 & 1292 & 399 & 1328 & \end{array}\right\} \therefore \text{NO.}$$

(7) Is 51 898 888 37 divisible by 467 ?

$\because$ **467×3=1401** $\therefore$ $Q_2=14$

$$\therefore \left\{\begin{array}{lllll} 51 & \overline{88} & 88 & \overline{88} & 37 \\ -467 & -37 & 504 & 430 & \end{array}\right\} \therefore \text{YES}$$

N.B. :—The alternative method of successive mechanical osculations is also, of course, available (but will prove generally less neat and tidy and will also be more tedious).

(8) Is 789405 35994 divisible by 647 ?

$\because$ 647×**17=10999** $\therefore$ $P_3=11$

$$\left\{\begin{array}{llll} 78 & 940 & 535 & 994 \\ 1294 & 6110 & 11469 & \end{array}\right\}$$

But **1294=647×2** $\therefore$ YES

(9) Is 2093 1726 7051 0192 divisible by 991 ?

$\because$ 991×111=110001 $\therefore$ $Q_4=11$.

$$\therefore \left\{\begin{array}{llll} \overline{2093} & 1726 & \overline{7051} & 0192 \\ -30721 & -526 & -4939 & \end{array}\right\}$$

But **30721=991×331** $\therefore$ YES

(10) Is 479466 54391 divisible by 421 ?

∵ 421 X 19=7999 ∴ P_3=8

$$\left\{\begin{matrix} 47 & 946 & 654 & 391 \\ 1694 & 7205 & 3782 & \end{matrix}\right\} \text{ NO}$$

(11) What change should be made in the first digit of the above number in order to render it divisible by 421 ?

Answer :—As 1684 is exactly 4×421, the only change needed in order to reduce the actually present 1694 into 1684 is the alternation of the first digit from 4 to 3.

CHAPTER XXXI

SUM AND DIFFERENCE OF SQUARES

Not only with regard to questions arising in connection with and arising out of Pythagoras' Theorem (which we shall shortly be taking up) but also in respect of matters relating to the three fundamental Trigonometrical-Ratio-relationships (as indicated by the three formulae $\sin^2\theta+\cos^2\theta=1$, $1+\tan^2\theta=\sec^2\theta$ and $1+\cot^2\theta=\operatorname{cosec}^2\theta$) etc., etc. we have often to deal with the difference of two square numbers, the addition of two square numbers etc., etc. And it is desirable to have the assistance of rules governing this subject and benefit by them.

Difference of two Square numbers

Of the two, this is much easier. For, *any* number can be expressed as the difference of two square numbers. The Algebraical principle involved is to be found in the elementary formula $a^2-b^2=(a+b)(a-b)$. This means that, if the given number can be expressed in the shape of the product of two numbers, our task is automatically finished. And this "if" imposes a condition which is very easy to fulfil. For, even if the given number is a prime number, even then it can be correctly described as the product of itself and of unity. Thus $7=7\times1$, $17=17\times1$, $197=197\times1$ and so on.

In the next place, we have the derived formula:

$(a+b)^2-(a-b)^2=4ab$; and therefore ab can always come into the picture as $\left(\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2$ = i.e. (half the sum)^2—(half the difference)^2; and as *any* number can be expressed as ab, the problem is readily solved. And the larger the number of factorisations possible, the better. In fact, if we accept fractions too as permissible, the number of possible solutions will be *literally infinite.*

For example, suppose we have to express 9 as the difference of two squares. We know that—

$$9=9\times1 \therefore 9=\left(\frac{9+1}{2}\right)^2-\left(\frac{9-1}{2}\right)^2=5^2-4^2 \text{ Similarly}$$

(1) $13\times1=(\frac{14}{2})^2-(\frac{12}{2})^2=7^2-6^2$

(2) $12=6\times2=4^2-2^2$

or $4\times3=3\frac{1}{2}^2-\frac{1}{2}^2$

or $12\times1=6\frac{1}{2}^2-5\frac{1}{2}^2$

(3) $48=8\times6=7^2-1^2$

or $12\times4=8^2-4^2$

or $16\times3=9\frac{1}{2}^2-6\frac{1}{2}^2$

or $48\times1=24\frac{1}{2}^2-23\frac{1}{2}^2$

or $24\times2=13^2-11^2$

The question, therefore, of expressing any number as the difference of two squares presents no difficulty at all !

The Sum of Two Square Numbers

Inasmuch, however, as a^2+b^2 has no such corresponding advantage or facilities etc., to offer, the problem of expressing any number as the sum of two square numbers is a tough one and needs very careful attention. This, therefore, we now proceed to deal with.

A Simple Rule in Operation

We first turn our attention to a certain simple rule at work in the world of numbers, in this respect.

We need not go into the relevant original Siitras and explain them (especially to our non-Sanskrit-knowing readers). Suffice it for us, for our present purpose, to explain their purport and their application.

Let us take a particular series of "mixed" fractions, namely,—$1\frac{1}{3}$, $2\frac{2}{5}$, $3\frac{3}{7}$, $4\frac{4}{9}$, $5\frac{5}{11}$ etc.
which fulfil three conditions:

(i) that the integer-portion consists of the natural numbers in order ;

(ii) that the numerators are exactly the same ; and

(iii) that the denominators are the odd numbers, in order, commencing from 3 and going right on.

It will be observed that, when all these fractions are put into shape as "improper" fractions,

i.e. as $\frac{4}{3}$, $\frac{12}{5}$, $\frac{24}{7}$, $\frac{40}{9}$, $\frac{60}{11}$ etc., etc.

the sum of D^2 and N^e is invariably equal to $(N+1)^2$! And the General Algebraical form being;,

$$n+\frac{n}{2n+1}=\left(\frac{2n\,(n+1)}{2n+1}\right)$$

$\therefore$ $D_n=2n+1$; and $N=2n\,(n+1)$

$\therefore$ $D^2+N^2=(N+1)^2$

or. $(2n+1)^2+4n^2(n+1)^2=(2n^2+2n+1)^2$

The *shape* of it is perhaps frightening; but the *thing* in itself is very simple: and the best formula is $D^2+N^2=(N+1)^2$.

This means that when a^2 (given)$+x^2$ is a perfect square, we can readily find out x^2. Thus, for instance,

(i) If a (the given number) be 9, $2n+1=9$ $\therefore$ $n=4$ $\therefore$ $4\frac{4}{9}$ is the fraction we want. And $9^2+40^2=41^2$

(ii) If a be 35, $2n+1=35$ $\therefore$ $n=17$ $\therefore$ The fraction wanted is $17\frac{17}{35}=\frac{612}{35}$ $\therefore$ $35^2+612^2=613^2$.

(iii) If $a=57$, $2n+1=57$ $\therefore$ $n=28$ $\therefore$ The required fraction is $28\frac{28}{57}=\frac{1624}{57}$ $\therefore$ $57^2+1624^2=1625^2$

(iv) If $a=141$, $2n+1=141$ $\therefore$ $n=70$ $\therefore$ The fraction wanted is $70\frac{70}{141}=\frac{9940}{141}$ $\therefore$ $141^2+9940^2=9941^2$

Note:—Multiples and sub-multiples too behave in exactly the same manner (according to *Ānurūpya* i.e. proportionately). For instance,

Let $a=35$ $\therefore$ $2n+1=35$ $\therefore$ $n=17$ $\therefore$ The fraction wanted is $17\frac{17}{35}=\frac{612}{35}$ $\therefore$ $35^2+612^2=613^2$

$\therefore$ $70^2+1224^2=1226^2$

A Simpler Method (for the same)

This same result can also be achieved by a simpler and easier method which does not necessitate the "mixed" fractions, the transforming of them into the "improper"—fraction—shape etc., but gives us the answer immediately.

It will be observed that, in all the examples dealt with above.

Since $D^2+N^2=(N+1)^2$

$\therefore D^2=(N+1)^2-N^2=2N+1=N+(N+1)$

In other words, the square of the given number is the sum of two consecutive integers at the exact middle. For instance, if 7 be the given number, its square=49 which can be split up into the two oonsecutive integers 24 and 25

$\therefore 7^2+24^2=25^2$. Similarly,

(1) If $a=9$, its square $(81)=40+41$ $\therefore 9^2+40^2=41^2$

(2) If $a=35$, its square $(1225)=612+613$ $\therefore 35^2+612^2=613^2$

(3) If $a=57$, its square $(3249)=1624+1625$ $\therefore 57^2+1624^2=1625^2$

(4) If $a=141$, its square $(19881)=9940+9941$ $\therefore 141^2+9940^2=9941^2$

and so on. And all the answers are exactly as we obtained before (by the first method).

The *Case* of *Even Numbers*

Yes; the square of an odd number is necessarily odd and can be split up into two consecutive integers. But what about even numbers whose squares will always be even and cannot be split up into two consecutive numbers? And the answer is that such cases should be divided off by 2 (and other powers of 2) until an odd number is reached and then the final result should be multiplied proportionately.

For example, if $a=52$, we divide it by 4 and get the odd number 13. Itssquare $(169)=84+85$ $\therefore 13^2+84^2=85^2$ $\therefore$ (multiplying all the terms by 4), we say: $52^2+336^2=340^2$.

There are many other simple and easy methods by which we can tackle the problem (of $a^2+b^2=c^2$) by means of clues and conclusions deducible from $3^2+4^2=5^2$, $5^2+12^2=13^2$, $8^2+15^2=17^2$ etc. But into details of these and other allied matters we do not now enter.

CHAPTER XXXII

ELEMENTARY SQUARING, CUBING ETC.

In some of the earliest chapters of this treatise, we have dealt, at length, with multiplication-devices of various sorts, and squaring, cubing etc., are only a particular application thereof. This is why this subject too found an integral place of its own in those earlier chapters (on multiplication).

And yet it so happens that the squaring, cubing etc., of numbers have a particular entity and individuality of their own ; and besides, they derive additional importance because of their intimate connection with the question of the square-root, the cube-root etc., (which we shall shortly be taking up). And, consequently, we shall now deal with this subject (of squaring, cubing etc.), mainly by way of Preliminary Revision and Recapitulation on the one hand and also by way of presentation of some important new material too on the other.

The Yāvadūnam Sutra (for Squaring)

In the revision part of it, we may just formally remind the student of the YĀVADŪNAM formula and merely cite some examples thereof as a sort of practical memory-refresher:

1. $97^2=94/09$;
2. $87^2=74,/_169=7569$
3. $19^2=1_1, 8_8\ 1=28_8\ 1=361$
4. $91^2=82/81$
5. $965^2=930_1/225=931/225$
6. $113^2=126_1/69=12769$
7. $996^2=992/016$
8. $998^2=996/004$
9. $9997^2=9994/0009$
10. $1007^2=1014/049$
11. $9996^2=9992/0016$
12. $9999^2=9998/0001$
13. $1017^2=1034/289$

14. $1039^2 = 1078_1/521 = 1079/521$
15. $99991^2 = 99982/00081$
16. $99998^2 = 99996/00004$
17. $99994^2 = 99988/00036$
18. $10004^2 = 10008/0016$
19. $999978^2 = 999956/000484$
20. $999998^2 = 999996/000004$
21. $100023^2 = 100046/00529$
22. $9999873^2 = 9999746/0016129$
23. $9999999^2 = 9999998/00000001$
24. $1000012^2 = 1000024/000144$

The Ānurūpya Sūtra (for Cubing)

This is new material. A simple example will, however, suffice to explain it:

Take the hypothetical case of one who knows only the cubes of the "first ten natural numbers" (i.e. 1 to 10) and wishes to go therebeyond, with the help of an intelligent principle and procedure. And suppose he desires to begin with 11^3.

1. The first thing one has to do herefor is to put down the cube of the first digit in a row (of 4 figures in a Geometrical Ratio in the exact proportion subsisting between them). Thus—

$11^3 =$ 1 1 1 1
       2 2
1 3 3 1

(ii) The second step is to put down, under the second and third numbers, just two tunes the said numbers themselves and add up. And that is all!

A few more instances will clarify the procedure:

(1) $12^3 =$ 1 2 4 8
       4 8
1 6 2 8
   1
1 7 2 8

(2) $13^3 =$ 1 3 9 27
       6 18
1 9 8 7
  2 2
2 1 9 7

(3) $$14^3 = \begin{array}{cccc} 1 & 4 & 16 & 64 \\ & 8 & 32 & \\ \hline 2 & 7 & 4 & 4 \\ \hline \end{array}$$

(4) $$15^3 = \begin{array}{cccc} 1 & 5 & 25 & 125 \\ & 10 & 50 & \\ \hline 3 & 3 & 7 & 5 \\ \hline \end{array}$$

(5) $$16^3 = \begin{array}{cccc} 1 & 6 & 36 & 216 \\ & 12 & 72 & \\ \hline 4 & 0 & 9 & 6 \\ \hline \end{array}$$

(6) $$17^3 = \begin{array}{cccc} 1 & 7 & 49 & 343 \\ & 14 & 98 & \\ \hline 4 & 9 & 1 & 3 \\ \hline \end{array}$$

(7) $$18^3 = \begin{array}{cccc} 1 & 8 & 64 & 512 \\ & 16 & 128 & \\ \hline 5 & 8 & 3 & 2 \\ \hline \end{array}$$

(8) $$19^3 = \begin{array}{cccc} 1 & 9 & 81 & 729 \\ & 18 & 162 & \\ \hline 6 & 8 & 5 & 9 \\ \hline \end{array}$$

(9) $$21^3 = \begin{array}{cccc} 8 & 4 & 2 & 1 \\ & 8 & 4 & \\ \hline 9 & 2 & 6 & 1 \\ \hline \end{array}$$

(10) $$22^3 = \begin{array}{cccc} 8 & 8 & 8 & 8 \\ & 16 & 16 & \\ \hline 10 & 6 & 4 & 8 \\ \hline \end{array}$$

(11) $$23^3 = \begin{array}{cccc} 8 & 12 & 18 & 27 \\ & 24 & 36 & \\ \hline 12 & 1 & 6 & 7 \\ \hline \end{array}$$

(12) $$24^3 = \begin{array}{cccc} 8 & 16 & 32 & 64 \\ & 32 & 64 & \\ \hline 13 & 8 & 2 & 4 \\ \hline \end{array}$$

(13) $$25^3 = \begin{array}{cccc} 8 & 20 & 50 & 125 \\ & 40 & 100 & \\ \hline 15 & 6 & 2 & 5 \\ \hline \end{array}$$

(14) $$32^3 = \begin{array}{cccc} 27 & 18 & 12 & 8 \\ & 36 & 24 & \\ \hline 32 & 7 & 6 & 8 \\ \hline \end{array}$$

(15) $$9^3 = (10-1)^3$$

$$\left.\begin{array}{l} = 1000-100+10-1 \\ \qquad\quad -200+20 \end{array}\right\} = 1000-300+30-1 = 729$$

(16) $$97^3 = \begin{array}{cccc} 729 & 567 & 441 & 343 \\ & 1134 & 882 & \\ \hline 912 & 6 & 7 & 3 \\ \hline \end{array}$$

or, better still, $97^3 = (100-3)^3$

$$\begin{array}{l} = 1000000-30000+900-27 \\ \qquad\qquad -60000+1800 \\ \hline \quad 1000000-90000+2700-27 \\ = \quad 912 \qquad\quad 6 \qquad 7 \quad 3 \\ \hline \end{array}$$

N.B.:—If you start with the cube of the first digit and take the next three numbers (in the top row) in a Geometrical

Proportion (in the ratio of the original digits themselves) you find that the 4th figure (on the right end) is just the cube of the second digit!

The Algebraical explanation hereof is very simple:

If a and b are the two digits, then our chart reads:

$$\begin{array}{r} a^3+a^2b+ab^2+b^3 \\ 2a^2b+2ab^2 \\ \hline a^3+3a^2b+3ab^2+b^3 \\ \hline \end{array}$$

and this is exactly $(a+b)^3$!

Almost every mathematical worker *knows* this; but very few people *apply* it! This is the whole tragedy and the pathos of the situation!

The Yāvadūnam Sūtra (for Cubing)

The same **YĀVADŪNAM Sūtra** can, in view of the above, be applied for cubing too. The only difference is that we take here not the deficit or the surplus but exactly twice the deficit or the surplus as the case may be and make a few corresponding alternations in the other portions also, as follows:

Suppose we wish to ascertain the cube of 104. Our base being 100, the excess is 4. So we add not 4 (as we did in the squaring operation) but double that (i.e. 8) and thus have 104+8 (= 112) as the left-hand-most portion of the cube. Thus we obtain 112.

Then we put down thrice the new excess multiplied by the original excess (i.e. 12×4 = 48) and put that down as the middle portion of the product. 112/48

And then we affix the cube of the original excess (i.e. 64) as the last portion thereof. And the answer is complete. 112/48/64

Some more illustrative instances are given below for familiarising the student with the new process (which is not *really* new but only a very useful practical application of the $(a+b)^3$ formula just above described:

(1) $103^3=10\dot{9}/27/27$ (because $9\times3=27$; and $3^3=27$)

(2) $113^3=139/07/97$ (because $39\times13=507$ and $13^3=2197$)
$\quad\quad\;\; 5\;21$
$=1442897$

(3) $1004^3=1012/048/064$ (because $12\times4=48$ and $4^3=64$

(4) $10005^3=\dot{1}0015/0075/0125$ (because $15\times5=75$ and $5^3=125$

(5) $996^3=988/048/\overline{064}=988/047/936$ ($\because$ $12\times4=48$ and $4^3=64$

(6) $93^3=79/47\;\;/\overline{43}$ (because $21\times-7=147$ and $-7^3=-343$
$\quad\quad\;\; 1\quad -3$
$=804357$

(7) $9991^3=9973/0243/\overline{0729}$ (because $-27\times-9=243$
and $-9^3=-729=9973/0242/9271$

(8) $10007^3=10021/0147/0343$

(9) $99999^3=99997/00003/0000\bar{1}=99997/00002/99999$

(10) $100012^3=100036/00432/01728$

(11) $99998^3=99994/00012/000\bar{8}=99994/00011/99992$

(12) $1000007^3=1000021/000147/000343$

(13) $999992^3=999976/000192/000512$ (because $24\times8=192$
& $8^3=512$

Fourth Power

We know that $(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4$. This gives us the requisite clue for raising any given number to its fourth power. Thus,

(1) $11^4 =$ 1 1 1 1 1
3 5 3
----
1 4 6 4 1

(2) $12^4 =$ 1 2 4 8 16
6 20 24
----
2 0 7 3 6

The *Binomial Theorem*

The "Binomial Theorem" is thus capable of practical application and—in its more comprehensive Vedic form—has thus been utilised, to splendid purpose, in the Vedic Sūtras. And a huge lot of Calculus work (both Differential and Integral) has been (and can bc) facilitated thereby. Bnt thcsc details, we shall hold over for a later stage.

CHAPTER XXXIII

STRAIGHT SQUARING

Reverting to the subject of the squaring of numbers, the student need hardly be reminded that the methods expounded and explained in an early chapter and even in the previous chapter are applicable only to special cases and that a General formula capable of universal application is still due.

And, as this is intimately connected with a procedure known as the *Dwandwa Yoga* (or the Duplex Combination Process) and as this is of still greater importance and utility at the next step on the ladder, namely, the easy and facile extraction of square roots, we now go on to a brief study of this procedure.

The *Dwandwa-Yoga* (*or* the *Duplex Combination* Process)

The term "*Dwandwa Yoga*" (or Duplex) is used in two different senses. The first one is by squaring ; and the second one is by Cross-multiplication. And, in the present context, it is used in both the senses (a^a and 2ab).

In the case of a single (central) digit, the square (a^a etc.,) is meant ; and in the case of a number of even digits (say, *a* and b equidistant from the two ends), double the cross-product (2ab) is meant.

A few examples will elucidate the procedure.

Denoting the Duplex with the symbol D, we have:

(1) For 2, $D=2^2=4$

(2) For 7, $D=49$

(3) For 34, $D=2\ (12)=24$

(4) For 74, $D=2\ (28)=56$

(5) For 409, $D=2\ (36)+0=72$

(6) For 071, $D=0+49=49$

(7) For 713, $D=2\ (21)+1^2=43$

(8) For 734, $D=2\ (28)+3^2=65$

(9) D for $7346 = 2 \times 42 + 2 \times 12 = 108$

(10) D for $26734 = 16 + 36 + 49 = 101$

(11) D for $60172 = 24 + 0 + 1^2 = 25$

(12) D for $73215 = 70 + 6 + 4 = 80$

(13) D for $80607 = 112 + 0 + 36 = 148$

(14) D for $77 = 2 \times 49 = 98$

(15) D for $521398 = 80 + 36 + 6 = 122$

(16) D for $746213 = 42 + 8 + 12 = 62$

(17) D for $12345679 = 18 + 28 + 36 + 20 = 102$

(18) D for $370415291 = 6 + 126 + 0 + 40 + 1 = 173$

(19) D for $432655897 = 56 + 54 + 32 + 60 + 25 = 227$

This is merely a recapitulation of the ***Ūrdhva Tiryak*** process of multiplication as applied to squaring and needs no' exposition.

Note :—If a number consists of n digits, its square must have *2n* or *2n-1* digits. So, in the following process, take extra dots to the left (one less than the number of digits in the given numbers).

Examples-

(1) $207^2 = \left.\begin{matrix} 40809 \\ 2\ 4 \end{matrix}\right\} = 42849$

```
  . . 207
 ---------
 4 0 809
   2  4
 ---------
 4 2 849
```

(2) $213^2 = 44_1369 = 45369$

(3) $221^2 = 48841$

(4) $334^2 = \begin{matrix} 9\ 8\ 3\ 4\ 6 \\ \ 1\ 3\ 2\ 1 \end{matrix} = 11556$

(5) $425^2 =$ . . 425
16/16/44/20/25 = 1806·25

(6) $543^2 = \begin{matrix} 25\ 0\ 6\ 49 \\ 4\ 4\ 2 \end{matrix} = 294849$

(7) 897^2 . . . . 8 9 7
64/144/193/126/49 = 80409

Or $1\ \bar{1}\ 0\ \bar{3}^2 = 1 - 2 + 1 - 6 + 6 + 0 + 9 = 80409$

Or (by *Yāvadūnam Sūtra*) $= 784/103^2 = 80409$

(8) $889^2 = \overset{.\ \ .\ \ 8\ \ 8\ \ 9}{\overline{64/128/208/144/81}} = 790321$

Or $\overline{1\ \bar{1}\ \bar{1}\ \bar{1}}^{\,2} = 0\ \ \overline{\underset{1-2-9+0+3+2+1}{0\ \ 0\ \ 1\ \ \bar{1}\ \ \bar{1}\ \ \bar{1}}} = 790321$

Or (by *Yāvadūnam Sūtra*)

$889^2 = 778/111^2 = 789/\ \ 321 = 790/321.$
$\qquad\qquad\qquad\qquad /12$

(9) $1113^2 = \overset{.\ \ .\ \ .\ \ 1\ \ 1\ \ 1\ \ 3}{\overline{1\ \ 2\ \ 3\ \ 8\ \ 7\ \ 6\ \ 9}}$

(10) $2134^2 = 4/4/13/22/17/24/16 = 4553956$

(11) $3214^2 = 9/12/10/28/17/8/16 = 10329796$

(12) $3247^2 = 9/12/28/58/46/56/49 = 10543009$

(13) $6703^2 = 36/84/49/36/42/0/9 = 44930209$

(14) $31\cdot42^2 = 9/6/25/20/20/16/4 = 987\cdot2164$

(15) $\cdot0731^2 = \cdot0049/42/23/6/1 = \cdot00534361$

(16) $8978^2 = 64/144/193/254/193/112/64 = 80604484$

Or $\overline{1\bar{1}02\bar{2}}^{\,2} = 1/\text{-}2/1\text{-}5/0/4/4/4 = 80604484$

Or (by *Yāvadūnam Sūtra*) $7956/1022^2 = 80604484$

(17) $8887^2 = 64/128/192/240/176/112/49 = 78978769$

Or $\overline{1\bar{1}\bar{1}\bar{1}3}^{\,2} = 1\text{-}2\text{-}10\text{-}3+8769 = 78978769$

Or (by *Yāvadūnam*) $7774/1113^2 = 7774/\ \ 8769 = 78978769$
$\qquad\qquad\qquad\qquad\qquad\qquad /123$

(18) $141.32^2 = 1/8/18/14/29/22/13/12/4 = 19971.3424$

(19) $21345^2 = 4/4/13/22/37/34/46/40/25 = 455609025$

(20) $43031^2 = 16/24/9/24/26/9/6/1 = 185 \times 66961$

(21) $46325^2 = 16/48/60/52/73/72/34/20/25 = 2146005625$

(22) $73\cdot214^2 = 49/42/37/26/70/28/17/8/16 = 5360329796$

CHAPTER XXXIV

VARGAMŪLA (SQUARE ROOT)

Armed with the recapitulation (in the last chapter) of the "Straight Squaring method" and the practical application of the *Dwandwayoga* (Duplex Process) thereto, we now proceed to deal with the *Vargamūla* (i.e. the Square Root) on the same kind of simple, easy and straight procedure as in the case of "Straight Division".

Well-known First Principles

The basic or fundamental rules governing the extraction of the square root, are as follows:

(1) The given number is fist arranged in two-digit groups (from right to left); and a single digit (if any) left over (at the left-hand-end) is counted as a simple group by itself.

(2) The number of digits in the square root will be the same as the number of digit-groups in the given number itself (including a single digit (if any such there be). Thus 16 will count as one group, 144 as two groups and 1024 as two.

(3) So, if the square root contains n digits, the square must consist of $2n$ or $2n-1$ digits.

(4) And, conversely, if the given number has n digits, the square root will contain $\frac{n}{2}$ or $\frac{n+1}{2}$ digits.

(5) But, in cases of pure decimals, the number of digits in the square is always double that in the square root.

(6) The squares of the first nine natural numbers are 1, 4, 9, 16, 25, 36, 49, 64 and 81. This means:

(i) that an exact square cannot end in 2, 3, 7 or 8;

(ii) *(a)* that a complete square ending in 1 must have either 1 or 9 (mutual complements from 10) as the last digit of its square root;

(*b*) that a square can end in 4, only if the square root ends in 2 or 8 (complements);

(c) that the ending of a square in 5 or 0 means that its square root too ends in 5 or 0 (respectively);

(d) that a square ending in 6 must have 4 or 6 (complements) as the last digit in its square root; and

(e) that the termination of an exact square in 9 is possible, only if the square root ends in 3 or 7 (complements).

In other words, this may be more briefly formulated thus:

(a) that 1, 5, 6 and 0 at the end of a number reproduce themselves as the last digits in its square;

(*b*) that squares of complements (from ten) have the same last digit. Thus, 1^2 acd 9^2; 2^2 and 8^2; 3^2 and 7^2; 4^2 & 6^2; 5^2 and 5^2; and 0^2 and 10^2 have the same ending (namely 1,4,9,6,5 and 0 respectively); and

(c) that 2,3,7 and 8 are out of court altogether, as the final digit of a perfect square.

Readily *Available First Data.*

Thus, before we begin the straight extracting of a square root by "straight division" method, we start with previous knowledge of (1) the number of digits in the square root and (2) the fist digit thereof. Thus—

(1) 74562814 N=8 ∴ N in square root=N/2=4; and the first digit thereof is 8.

(2) 963106713. N=9 ∴ Nin the square root=$\frac{N+1}{2}$=5; & the first digit thereof is 3.

But (3) $(\cdot 7104)^2$ *must* contain 8 decimal digits.

(4) $\sqrt{\cdot 16}=\cdot 4$

(5) $\sqrt{\cdot 0064}=\cdot 08$

(6) $\sqrt{\cdot 000049}=007$

(7) $\sqrt{\cdot 00007(0)} = \cdot 008$ etc.

(8) $\sqrt{\cdot 00000007} = \cdot 0002$ etc.

(9) $\sqrt{\cdot 09} = \cdot 3$

But (10) $\sqrt{\cdot 9} = \sqrt{\cdot 90} = \cdot 9$ etc.

Modus operandi (of *Straight* Squaring)

The procedure of Straight Squaring as inculcated in the Vedic Sūtras is precisely the same as in Straight Division but with this difference,namely, that in the former the Divisor should be excatly double the first digit of the square root.

N.B. As a single digit can never be more than 9, it follows therefore that, in our method of straight squaring, no divisor above 18 is necessary. We may, of course, ***voluntarily choose*** to deal with larger numbers ; but there is no ***need*** to do so.

Initial Chart

We thus start our operation with an initial chart, like the samples given hereunder :

(1)
```
     5 : 29 :
4 :    : 1  :
-------------
   : 2 :
```

(2)
```
     7 : 31 :
       : 3  :
-------------
4 :  2 :
```

(3)
```
     32 : 49 :
      : : 7  :
--------------
10 : 5 :
```

(4)
```
     40 : 96 :
   :    : 4  :
--------------
12 :  6 :
```

(5)
```
     1 : 63 84 :
2 :    : 1
----------------
   : 1 :
```

(6)
```
     8 : 3176 :
4 :    : 4
---------------
     2 :
```

(7)
```
     44 : 44 44 :
12 :    : 8
-----------------
   : 6 :
```

(8)
```
     61 : 136 :
14 :    : 12  :
---------------
   : 7 :
```

(9)
```
     73 : 60 : 84 :
16 :    : 9  :    :
-------------------
   : 8 :     :    :
```

(10)
```
     6 : 00 00 01 :
4 :    : 2
-------------------
   : 2 :
```

```
(11)     10 :  73 69 42 :        (12)      90 :  61 71 74 :
       6 :    : 1          :            18 :     : 9         :
         ----------------                 -----------------
         :  3 :            :              :  9 :             :
         ----------------                 -----------------
```

Further Procedure

Let us now take a concrete case (the extraction of the square root of, say, 119716) and deal with it:

(i) In the above given general chart, we have not only put down the single first digit of the square root wanted but also prefixed to the next dividend-digit, the remainder after our subtraction of the square of that first digit from the left-hand-most digit or digit-group of the given number. And we have also set down as our divisor, the exact double of the first digit of the quotient.

```
     11 :  9 7 1 6 :
  6 :    : 2
    ----------------
    :  3 :
    ----------------
```

(ii) Our next Gross Dividend-unit is thus 29. Without subtracting anything from it, we simply divide the 29 by the divisor 6 and put down the second quotient-digit 4 and the second remainder 5 in their proper places as usual.

```
  6 : 11 : 2 9  5 7  1   6 :
    ----------------
    :  3 : 4
    ----------------
```

(iii) Thus our third Gross Dividend is 57. From this we subtract 16 (the square of the second quotient-digit), get 41 as the Actual Dividend, divide it by 6 and set down the Q (6) and the R (5) in their proper places as usual.

```
     11 :  9 7 1 6 :
  6 :    : 2 5 5   :
    ----------------
    :  3 : 4 6
    ----------------
```

(iv) Our third gross dividend-unit is 51. From this we subtract the *Dwandwa* Yoga (Duplex) (=48), obtain 3 as the remainder, divide it by 6 and put down the Q (0) and the R (3) in their proper places.

```
      11 : 9716
  6 :    : 2553
    ----------------
    :  3 : 46.00
    ----------------
```

(v) This gives us 36 as our last gross dividend-unit. From this we subtract 36 (the *Dwandwa Yoga* of the third quotient digit 6); get 0 as Q and as R. This means that the work has been completed, that the given expression is a perfect square and that 346 is ita square root. And that is all.

	11 :	9 7 1 6
6:		: 2 5 5 3
	: 3 :	4 6 . 0 0

COMPLETE

Proof of Completeness & Correctness

(1) A manifest proof of the Complete-squareness of the given expression (and of the correctness of the square root ascertained) is by squaring the latter and finding the square to be exactly the same as the given complete square. Thus,

346^2—9/24/52/48/36=119716 .. YES.

(2) But this is too mechanical. We obtain a neat and valid proof from the very faet that, if and when the process is continued into the decimal pert, all the quotient-digits (in the decimal part) are found to be zeroes and the remainders too are all zeroes!

Proof to *the Contrary*

A number can *not* be an exact square in the following circumstances:

(1) if it ends in 2, 3, 7 or 8;

(2) if it terminates in an odd number of zeroes;

(3) if its last digit is 6 but its penultimate digit is even;

(4) if its last digit is not 6 but its penultimate digit is odd; and

(5) if, even though the number be even, ita last two digits (taken together) are not divisible by 4;

AND a square root cannot be correct if it falls to fulfil any of the requirements hereinabove indicated:

Examples

Some instructive illustrative examples are given below:

(1)
```
      5 : 2 9   :
  4 :   : 1 0
  -----------
  :  2 : 3 (complete):
```

(2)
```
     32 : 4 9
 10 :    : 7 4
 -------------
 : 5 : 7 (complete:
```

(3)
```
      40 : 9 6   :
 12 :    : 4 1
 -------------
 :  6 : 4 (complete):
```

(4)
```
      1 : 6 3 84
  2 :   : 0 2 3 6
  ---------------
  : 1 : 2  8 (complete):
```

(5)
```
      : 55 : 2 0 4 9
 14 :      : 6 6 2
 -------------------
  :  7 :  4 3.00 (complete)
```

(6)
```
      : 69 : 9  0 6 3 1
 16 :      : 5 11 5 41
 ----------------------
  : 8 : 3 6 1. 0 0 0
```

∴ A complete square

(7)
```
 14 : 53 : 1  6 3  2 1 4
  :        : 4 13 6 13 5 7
 --------------------------
  :  7 : 2 9 1 ·3 1...(incomplete)
```

(8)
```
 6 : 14 : 0 4  7  5 0 4
 :        : 5 8 11 13
 ----------------------
 :  3 :  7 4 8 ...(incomplete)
```

(9)
```
 12 : 41 : 2 5 4 9 2 9
  :        : 5 4 5 2 1 0
 ------------------------
  :  6 : 4 2 3 ·0 0 0
```

∴ A complete square

(10)
```
 4 : 7 : 3 8 9  1 5 4   8 0
 :     : 3 5 5 13 6   7 4   0
 ----------------------------
 :  2 :  7 1  8 3 ·0000
```

∴ An exact square

(11)
```
 10 : 25 : 7 4 5 4 7 6
  :        : 0 7 4 5 5 1
 ------------------------
  :  5 : 0 7 4'0 0 0
```

∴ A perfect square

(12)
```
 12 : 45 : 3 1 9 8 2 4
  :        : 9 9 6 3 1
 ----------------------
  :  6 :  7 3 2.0 0 0
```

∴ A complete square

(13) 16 : 74 : 5 7 5 3 1 4 49
: : 10 9 13 19 12 7 7 4

: 8 : 6 3 5 7 · 0 0 0 ∴ An exact square

(14) 14 : 52 : 4 4 3 9 0 7 (to 2 places of decimals)
: : 3 6 4 13 7

: 7 : 2 4 1 · 80 ...

(15) 16 : 73 : 2 1 0 8 (to 3 decimal-places)
: : 9 12 16 14 15

: 8 : 5 5· 6 3 3 ...

(16) 8 : 18 : 1 3 4 5 1 2 6 (to 3 decimal-places)
: 2 5 9 10 1617

: 4 : 2 ·5 8 4 6 4

(17) 6 : 13 : 8 7 0 0 0 (to 4 decimal-places)
: : 4 6 6 8 8 12

: 3 : 7 '2424 ...

(18) 16 : 75 : 0 1 7 1 0 0 0 0 (to 4 places)
: : 11 14 9 11 10

: 8 : 6 6 1 24 ...

(19) 6 : '00 : 09 : 2 4 0 1 6 0 (to 5 places)
: : : 0 2 6 6

: ·0 : 3 : 0 3 9 7 ...

(20) 8 : 16· : 7 9 0 0 0 0 0 0 0 (to 5 places)
: : 0 7 7 14 19

: 4· : 0 9 7 5 6 ...

(21) 10 : 27· : 1 3 0 0 0 0 0 0 0 0 0 (to 5 places)
: : 2 1 9 10 8 16 7

: 5 · : 2 0 8 64 ...

(22) 16 : ·74 : 1 0 7 0 0 000 (to 5 places)
: : 10 5 14 19 14

: ·8 : 6 0 8 50 ...

(23) 8 : 19· : 7 0 6 4 1 2 8 14 (to 6 places)
: : 3 5 10 10 15 17 13

:4· : 4 3 9 1 9 0 · .

(24) 10 : 27· : 0 0 0 0 0 0 0 0 (to 6 places)
: : 2 10 9 12 17 10

:5· : 1 9 6 1 5 2 4 ...

(25) 6 : ·09 : 0 0 4 6 13 (to 5 decimal-places)
: : 0 0 0 4 3 1

: ·3 : 0 0 07 5 ...

(26) 6 : ·0009 : 1 3 4 0 0 0 0 0 (to 6 places)
: 0 1 1 2

: ·03 : 0 2 2 2 ...

(27) 12 : '0039 : 3 0 0 0 0 0 0 0 (to eight places)
: 3 9 14

: ·06 : 2 6 8 9 7 1 ...

(28) 18 : 00000083 : 1 0 0 0 0 0 0 0 (to 8 places)
: 2 3 11 18 7 2 0

: '00 0 9 : 11 5 9 ...

(29) 18 : ·000092 : 4 0 1 0 0 0 0 0 (to ten places)
: 11 6 6 13 15 14 24 32

: ·0 09 : 6 1 2 5 4 38 ...

(30) 2 : 2 : 0 7 3 6
: : 1 2 3 1

: 1 : 4 4. 0 0 ∴ A complete square.

Or, taking the first two digits together at the first step, we have :

28 : 207 : 3 6
: : 11 1

: 14 : 4. 0 ∴ An exact square

CHAPTER XXXV

CUBE ROOTS
of
EXACT CUBES

(Mainly by Inspection and Argumentation)

(*Well-known*) FIRST-PRINCIPLES

(1) The lowest cubes (i.e. the cubes of the first nine natural numbers) are 1, 8, 27, 64, 125, 216, 343, 512 and 729.

(2) Thus, they all have their own distinct endings; and the is no possibility of over-lapping (or doubt as in the case of squares).

(3) Therefore, the last digit of the cube root of an exact cube is obvious:

(i) Cube ends in 1; ∴ cube root ends in 1;
(ii) C ends in 2; ∴ C R ends in 8;
(iii) C ends in 3; ∴ C R ends in 7;
(iv) C ends in 4; ∴ C R ends in 4;
(v) C ends in 5; ∴ C R ends in 5;
(vi) C ends in 6; ∴ C R ends in 6;
(vii) C ends in 7; ∴ C R ends in 3;
(viii) C ends in 8; ∴ C R ends in 2; and
(ix) C ends in 9; ∴ C R ends in 9;

(4) In other words,

(i) 1, 4, 5, 6, 9 and 0 repeat themselves in the cube-endings; and

(ii) 2, 3, 7 and 8 have an inter-play of complements (from 10).

(5) The number of digits in a cube root is the same as the number of 3-digit groups (in the original cube) including a single digit or a double-digit group (if any such there be).

(6) The first digit of the cube-root will always be obvious (from the first group in the cube).

(7) Thus, the number of digits, the first digit and the last digit of the cube root of an exact cube are the data with which we start, when we enter on the work of extracting the cube root of an exact cube.

Examples

(Let a, 1 and n be the symbols for the first digit, the last digit and n the number of digits in the cube root of an exact cube).

(I) For 271, 601, **f=6, 1=1** and n=2
(2) For **4**, 269, 813, F=1, **L=7** and **n=3**
(3) For 5, 678, **f=1, L=2** and n=2
(4) For 33,076, 161, **F=3,** L=1 and n=3
(5) For 83,453,453, F=4, L=7 and n=3
(6) For 105, 823, 817, **f=4,** L=3 **and n=3**
(7) For 248, 858, 189, **f=6,** L=9 and n=3
(8) For 1, 548, 816, 893, **f=1, L=7** and n=4
(9) For 73,451, 930, 798, **f=4, L=2** and n=4
(10) For 76, 928, 302, 277, **f=4, L=3** and n=4
(11) For 6, 700, 108,456, 013, f=1, L=7 and n=5
(12) For 62, 741, 116, 007, 421, **f=3,** L=1 and n=5
(13) For 91, 010, 000, 000, 468, **f=4, L=2** and **n=5** and so on.

The Chart-Preliminary and Procedure

The **procedure** is similar to the one adopted by us in "Straight Division" and particularly in the extraction of square roots. The only difference is that our divisor (in this context) will not be double the first digit of the root but thrice the ***square*** thereof. As we know the first digit at the very outset, our chart begins functioning as usual (as follows):

(1) **3** : 1 : 7 2 8
: : **0 1**
: 1 :

(2) 12: 13: 8 2 4
: : **5**
: 2:

```
(3)  48 : 73 :  089149        (4) 27 : 27 :  8 4 1
        :     : 901                  :     : 0
        -----------------            -----------------
        :  4 :                       :  3 :
        -----------------            -----------------

(5)  12 : 21 :   400 713      (6) 192 : 600 :   132 419
        :     : 13                   :       : 88
        -----------------            -----------------
        :  2 :                       :   8 :
        -----------------            -----------------

(7)  48 : 79 :   314 502
        :     : 15
        -----------------
        :  4 :
        -----------------
```

Algebraical Principle Utilised

Any arithmetical number can be put into its proper algebraical shape as:

$a+10b+100c+1000d$ etc.

Suppose we have to find the cube of a three-digit arithmetical number. Algebraically, we have to expand $(a+10b+100c)^3$. Expanding it accordingly, we have:

$$(a+10b+100c)^3=a^3+100b^3+1000000c^3+30a^2b+300ab^2$$
$$+300a^2c+30000ac^2+300000b^2c+300000bc^2+6000abc.$$

Removing the powers of ten and putting the result in algebraical form, we note the following:

(1) The units' place is determined by a^3.

(2) The tens' place is determined by $3a^2b$

(3) The hundreds' place is contributed to by $3ab^2+3a^2c$

(4) The thousands' place is formed by b^3+6abc

(5) The ten thousands' place is given by $3ac^2+3b^2c$

(6) The lakhs' place is constituted of $3bc^2$; and

(7) The millions' place is formed by c^3.

N.B.:—The number of zeroes in the various coefficients (in the Algebraical expansion) will prove the correctness of this analysis.

Note:—If one wishes to proceed in the reverse direction, one may do so; and, for facility's sake, the letters substituted (for a, b, c, d etc.) may be conveniently put down as L, K, J, H etc.

The Implications of the Principle

This Analytical sorting of the various parts of the algebraical expansion into their respective places, gives us the necessary clue for eliminating letter after letter and determining the previous digit. And the whole procedure is really of an argumentational character. Thus,

(i) Prom the units' place, we subtract a^3 (or L^3); and that eliminates the last digit.

(ii) From the ten's place, we subtract $3a^2b$ (or $3L^2K$) and thus eliminate the penultimate digit.

(iii) From the hundreds' place, we subtract $3a^2c+3ab^2$ (or 2 $3L^2J+31k^2$) and there-by eliminate the pre-penultimate digit.

(iv) From the thousands' place, we deduct $b3+6abc$; and so on

N.B.:—In the case of perfect cubes we have the additional advantage of knowing the last digit too, beforehand.

Some instructive examples are given below:

(1) Extract the cube root of the exact cube 33, 076, 161.

Here $a=3$; $L=1$; and $n=3$.

(L) $L=1$ ∴ $L^3=1$.

∴ Subtracting 1, we have

(K) $3L^aK=3k$ (ending in 6) ∴ K=2

Deducting 3K; we have

(J) $3L^2J+3LK^2=3J+12$ (ending in 1)

∴ 3J ends in 9 ∴ J=3

$$\begin{array}{r} 1 \quad 1 \quad \\ 33\ 076\ 161 \\ 1 \\ \hline 33\ 076\ 16 \\ 6 \\ \hline 33\ 0761 \end{array}$$

∴ CR=321

N.B.:—The last step is really unnecessary (as the first digit is known to us from the outset).

(2) Extract the cube root of the exact cube 1728.

Hero, a=1 ; L=2 and n=2 ∴ CR=12

(3) Extract the cube root of the exact cube 13,824

Here a=2 ; L=4 ; and n=2 ∴ CR=24

(4) Determine the cube root of the exact cube 83,453,453.

Here F=4 ; L=7 ; and N=3

(L) L=7 ∴ L^3=343.
Subtracting this, we have

83 453 453
343
83 453 11

(K) $3L^2K$=147K (ending in 1) ∴ K=3
∴ subtracting 441

441
83 448 7

(J) $3L^2J+3LK^2$=147J+189 (ending in 7)
∴ 147J ends in 8 ∴ J=4 ∴ CR=437

N.B.:—Exactly as in the previous example.

(5) Find out the cube root of the exact cube 84, 604, 519

Here a=4 ; L=9 ; and n=3

(L) L=9 ∴ L^3=729 ∴ Subtracting this

84 604 519
729
8460379

(K) $3L^2K$=243K (ending in 9) ∴ K=3
Subtracting 729,

729
845965

(J) $3L^2J+3LK^2$=243J+243 (ending in 5)
∴ 243J ends in 2 ∴ J=4 ∴ CR=439

N.B.:—As before.

(6) Extract the cube root of the exact cube 2488 58189

Here a=6 ; L=9 ; and n=3.

2488 58189
729
24885 746

(L) L=9 ∴ L^3=729 ∴ Subtracting this,

(K) ∴ $3L^2K$=243K (ending in 6)
∴ K=2 ∴ Deducting 486)

486
248852 6

(J) $3L^2J+3LK^2$=243J+108 (ending in 6)
∴ 243J ends in 6 ∴ J=6) CR=629.

N.B. Same as before.

(7) Determine the cube root of the exact cube 105823817

Here a=4 ; L=3 ; and n=3

(L) L=3 ∴ L^3=27. Subtrarting this

105823817
27
10582379

(K) $3L^2K$=27K (ending in 9) ∴ K=7
Subtracting 189, we have

189
1058219

(J) $3L^2J+3LK^2=27J+441$ (ending in 8) } $\therefore$ CR=473
$\therefore$ J=4

N.B. As before,

(8) Extract the cube root of the exact cube 143 055 667

Here a=5 ; L=3 ; and n=3

143 055 667

(L) L=3 $\therefore L^3=27$. deducting this

143 055 64

(K) $3L^2K=27K$ (ending in 4) $\therefore$ K=2

Subtracting 54, we have

54

143 055 1

(J) $3L^2J+3LK^2=27J+36$ (ending in 1) } . CR=523
$\therefore$ J=5

N.B. Exactly as before.

(9) Bind the cube root of the cube 76,928, 302,277.

Here a=4, L=3 ;
and n=4.

The last 4 digis are 2277

27

225

(L) L=3 ... $L^3=27$. Subtracting this,

(K) $3L^2K=27K$ (ending in 5) }
$\therefore$ K=5 $\therefore$ Subtracting 135_1

135

09

(J) $3L^2J+3LK^2=27J+225$ (ending in 9) $\therefore$ J=2 } $\therefore$ CR=4253

N.B. :—But, if, on principle, we wish to determine the first digit by the same method of successive elimination of the digits, we shall have to make use of another algebraical expansion (namely, of $(L+K+J+H)^3$. And, on analysing its parts as before into the units, the tens, the hundreds etc., we shall find that the 4th step will reveal $3L^2H+6LKJ+K^3$ as the portion to be deducted. So,

(H) $3L^2H+6\,LKJ+K^S=27H+180$
$+125=27H+305$ (ending in 3)
$\therefore$ H=4 : and CR=4253

30 : 09
2 : 79
27 : 3

(10) Determine the cube mot of the cube 11,345, 123, 223

Here a=2 ; L=7 : and n=4

11 345 123 223

(L) L=7 $\therefore L^3=343$. Subtracting it,

343

11 345 12 288

(K) $3L^2K+147K$ (ending in 8) ∴ K=4. Deducting 588

$$\begin{array}{r} 688 \\ \hline 11\ 345\ 11\ 70 \end{array}$$

(J) $3LJ^8+3LK^g=147J+336$ (ending in 0) ∴ J=2 ∴ Subtracting 330, we have

$$\begin{array}{r} 330 \\ \hline 1134508\ \ 4 \end{array}$$

(H) $3L^aH+6\ LKJ+K^3=147H+336+64=147H+400$ (ending in 4). ∴ H=2

∴ **CR=2247**

N.B. The last part (for ascertaining the first digit) is really superfluous.

(11) Extract the cube root of the cube 12,278,428,443

Here a=2 ; L=7 ; and n=4.

(L) L=7 ∴ $L^3=343$ ∴ Deducting it

$$\begin{array}{r} 12\ 278\ 428\ 443 \\ 343 \\ \hline 12\ 278\ 428\ 10 \end{array}$$

(K) $3L^2K=147K$ (ending in 0) ∴ K=0.

$$\begin{array}{r} 0 \\ \hline 12\ 278\ 428\ 1 \end{array}$$

(J) $3L^2J+3LK^2=147J+0$ (ending in 1) ∴ J=3

$$\begin{array}{r} 441 \\ \hline 12\ 278\ 384 \end{array}$$

∴ Subtracting 441, we have

(H) $3L^2H+6LKJ+K^3=147H+0+0$ and ends in 4 ∴ H=2

∴ **CR=2307**

N.B. As in the last example.

(12) Find the cube root of the cube 355 045 312 441

Here a=7 ; L=1 ; n=4

(L) L=1 ∴ $L^3=1$. Deducting it,

$$\begin{array}{r} 355045\ 312\ 441 \\ 1 \\ \hline 355045312\ 44 \end{array}$$

(K) $3L^2K=3K$ (ending in 4) ∴ K=8 ∴ Deducting 24.

$$\begin{array}{r} 24 \\ \hline 3550453122 \end{array}$$

(J) $3L^2J+3\ LK^2=3J+192$ (ending in 2) ∴ J=0

$$\begin{array}{r} 192 \\ \hline 355045293 \end{array}$$

(H) $3L^2H+6LKJ+K^3=3h+0+512$ and ends in 3 ∴ h=7

∴ **CR=7081**

N.B. Exactly as above.

Note:—The above method is adapted mainly for odd cubes. If the cube be even, ambiguous values may arise at each step and tend to confuse the student's mind.

(13) Determine the cube root of the cube 792 994219216

Here a=9 ; L=6 ; and n=4

(L) L=6 ∴ L^3=216. Deducting this,

792.994 219 216
216
~~792 994 219 00~~

(K) $3L^2K$=108K (ending in zero)
∴ K=0 or 5. Which is it to be ?

540
772 994 2136

Let us take 5 (a pure gamble) !

(J) $3L^2J+3LK^2$=108J+450 (ending in 6) ∴ J=2 or 7 !
Which should we prefer ? Let us accept 2 (another perfect gamble) !

666
7729961 47

(H) $3L^2H+6LKJ+K^3$—108H+360 +125=108H+485(ending in 7)
∴ H=4 or 9 ! Which should we choose ? Let us gamble again and pitch for 9 !

∴ CR=9256

Here, however, our previous knowledge of the first digit may come to our rescue and assure us of its being 9. But the other two were pure gambles and would mean 2×2 i.e. four different possibilities !

A Better Method

At every step, however, the ambiguity *can* be removed by proper and cogent argumentation ; and this may also prove *interesting*. And anything intellectual may be welcomed ; but it should not become too stiff and abstract ; and an ambiguity (in such a matter) is wholly *undesirable* (to put it mildly). A better method is therefore necessary, is available and is given below.

All that has to be done is to go on dividing by 8 (until an odd cube emanates), work the sum out and multiply by the proper multiplier thereafter. Thus,

8) 792 994 249 216

8) 99124 281152

8) 12390535144.

1 548 816 893
1 1 1

Here $a=1$; $L=7$; and $n=4$

(L) $L=7 \quad \therefore L^3=343$. Subtracting this,

1648816893
343
154881656

(K) $3L^2K=147K$ (ending in 5) $\therefore K=5$. Deducting 735.

736
15488092
672

(J) $3L^2J+3LK^2=147J+525$ (endingin 2) $\therefore J=1$ Deducting 672, we have : 1548 742

(H) $3L^2H+6LKJ+K^3=147K+210+125=147H+335$ (ending in 2) $\therefore H=1$

$\therefore$ The cube root is 1157 And

and $\therefore$ CR (of the original cube$=8\times1157=9256$

N.B. Here too, the last step is unnecessary (as the first digit is already known to us).

(14) Determine the cube rot of the cube 2, 840, 362, 499, 528

Here $a=1$; $L=2$; and $n=5$

(L) $L=2 \therefore L^a=8$

2840 362 499 528
8

$\therefore$ Deducting this, we have 2840 362 499 52

(K) $3L^2K=12K$ and ends in 2 $\therefore K=1$ or 6 ! Let us take 6 ! Deducting 12K.

72
2840 362 4988
228

(J) $3L^2J3LK^2=12J+216$ (ending in 8) $\therefore J=1$ or 6 ! Let us take 1 !

2840 362 476

(H) $3L^2H+LKJ+K^3=12H+12+216=2H+238$ (ending in 6) $\therefore H=4$ or 9 : Let us take 4 !

(G) We need not bother ourselves about G and the expansion of $(a+b+c+d+e)^3$ and so on. Obviously $G=1$

$\therefore$ CR$=14162$

But the middle three digits have been the subject of uncertainty (with $2\times2\times2=8$ different possibilities). We

must therefore work this case too out by the other — the unambiguous — method.

or 8 : 2 8 4 0 3 6 2 4 9 9 5 2 8

3 5 5, 0 4 5, 3 1 2, 4 4 1

Here a=7 ; L=1 ; and n=4

(L) L=1 ∴ L^3=1
∴ Subtracting this, we have

365 045 312 441
1
355 045 312 44

(K) $3L^2K$=3K (ending in 4)
∴ K=8 ∴ Subtracting 24,

24
355 045 312 2

(J) $3L^2J+3LK^2$=3J+192
and ends in 2 ∴ J=0
∴ Subtracting 192,

19 2
355 045 13

(H) $3L^2H+6LKJ+K^3$=3H+0+512
and ends in 8 ∴ H=7 ∴ Cube Root=7081

CR of the original expression=14162

(15) Find out the 12-digit exact cube whose last four digits are 6741.

Here a= ? ; L=1 ; and n=4 . . . 6 7 4 1

(L) L=1 ∴ L^3=1 ∴ Subtractingit,

1
. . . 6 7 4

(K) $3L^2K$=3K and ends in 4
K=8 ∴ Deducting 24.

2 4
. . . 6 5

(J) $3L^2J+3LK^2$=3J+192 and ends in 5 ∴ J=1
∴ Subtracting 195, we have

. . 1 9 5
. . . 7

(H) $3L^2H$+6 LKJ+K^3=3H+48+512=3H+560 and ends in 7.
∴ H=9

∴ The original Cube Root is 9181

N.B. As we did not know the first digit beforehand, all the steps were really necessary.

(16) A 13-digit perfect cube begins with 5 and ends with 0541. Find it and its cube root.

Here a=1 ; **L=1** ; and **n=5.** **0 5 4 1**

(L) **L=1** ∴ $L^3=1$. Deducting it. 1

0 5 4

(K) $3L^2K=3K$ ∴ K=8 } 2 4

∴ Subtracting 24, we have **0 3**

(J) $3L^2J+3LK^2=3J+192$ } 2 1 3

and ends in **3** ∴ **J=7** } . . . **9**

∴ Deducting 213, we have

(H) $3L^2H+6LKJ+K^3=3H+336$

$+512=3H+848$ and ends in 9 }

∴ **H=7**

(J) And **G=1** ∴ CR= 17781

And the cube= 17781^3

CHAPTER XXXVI

CUBE ROOTS (GENERAL)

Having explained an interesting method by which the cube roots of exact cubes can be extracted, we now proceed to deal with cubes *in general* (i.e. whether exact cubes or not). As all numbers cannot be perfect cubes, it stands to reason that there should be a general provision made for *all* cases. This, of course, there is; and this we now take up.

First Principles

It goes without saying that all the basic principles explained and utilised in the previous chapter should hold good here too. We need not, therefore, re-iterate all that portion of the last chapter but may just, by way of recapitulation, remind ourselves of the *conclusions* arrived at there and the *modus operandi* in question.

The Sequence of the Various Digits

(1) The first place by a^3
(2) The second place by $3a^2b$
(3) The third place by $3ab^2+3a^3c$
(4) The fourth place by $6abc+b^3$
(5) The fifth place by $3ac^2+3b^2c$
(6) The sixth place by $3bc^2$
(7) The seventh place by c^3; and so on.

The Dividends, Qotients, and Remainders

(1) The first D, Q and R are available at sight.
(2) From the second dividend, no deduction is to be made.
(3) From the third, subtract $3ab^2$
(4) From the fourth, deduct $6\ abc+b^3$
(5) from the fifth, subtract $3ac^2+3b^2c$
(6) from the sixth, deduct $3bc^2$
(7) from the seventh, subtract c^3. ; and so on.

Modus Operandi

Let us take a concrete example, namely, 258 474 853 and see the *modus operandi* actually at work (step by step)

```
     258 :  4    74853
108 :     : 42 100
    ------------------
     : 6 : 3
```

(a) Put down 6 and 42 as first Q and first R by mere *Vilokanam* (inspection).

(b) The second Gross Divident is thus 424. Don't subtract any thing therefrom. Merely divide it by 108 and put down 3 and 100 as Q, and R_2

```
    : 258 :  4     74853
108 :     : 42 100
    ---------------------
    :   6 :  3  7
```

(c) So, the third Gorss Dividend is 1007. Subtract therefrom $3ab^2$ (i.e. $3 \times 6 \times 3^2$ i.e. 162). The third Actual Working Dividend therefore is $1007-162=845$. Divide this by 108 and set down 7 and 89 as Q_3 and R_3

```
    : 258 :  4   7    4853
108 :     : 42 100 89
    ----------------------
    :  6  :  37
```

(*d*) Thus, the fourth Gross Dividend is 894. Subtract therefrom $6\,abc+c^3=(756+27=783)$.

```
       258 :  4    7   4   853
108 :      : 42  100  89  111
    ---------------------------
    :  6 :  37    0
```

So, the fourth actual working dividend is $894-783=111$. Divide this again by 108 and put down 0 and 111 as Q_4 and R_4.

(e) Our next gross dividend is now 1118. Subtract therefrom $3ac^2+3b^2c=882+189=1071$. Therefore our fifth actual working dividend is 47. Divide it by 108 and put down 0 and 47 as Q_5 and R_5

```
      258 : 4  7  4   8  5  3
108 :     : 42 100 89 111 47
    ---------------------------
    : 6 : 3  7  0  0   0
```

(f) Our sixth gross dividend is **475.** Subtract therefrom $3bc^2$ (=441) So, our Q, and R, now are 0 and 34.

	258 :	4	7	4	8	5	3
108 :		:42/	100/	89/	111/	47/	34
	: 6 :	3	7	0	0	0	

(complete cube)

(g) Our last gross dividend is thus **343.** Subtract C^3 (=343) therefrom and set down 0 and 0 as our Q, and 6,

This means that the given number is a perfect cube, that the work (of extracting its cube root) is over and that the cube root is **637.**

N.B. Proof of the correctness of our answer is, of course, readily available in the shape of the fact that 637^3 is the given number. But this will be too crudely and cruelly laborious. Sufficient proof, however, is afforded by the very fact that, on going into the decimal part of the answers, we find that all the quotients and all the remainder sare zeroes.

An Incomplete cube is now dealt with as a sample :

Extract the cube root of 417 to 3 places of decimals

Here the divisor is 147.

	417· :	0	0	0	0
147 :	3 43 :	74	152	155	163
	7		4	7	1

(*a*) Here Q_1 and R_1=7 and 74

(*b*) ∴ The second gross dividend is **740.** No subtraction is required. ∴ Dividing 740 by 147, we get 4 and 152 as Q_2 and R_2.

(c) ∴ The third gross dividend is **1520.** Subtracting $3ab^2$ (=336) therefrom, we have 1184 as our third actual working dividend. We divide it by 147 and put down 7 and 155 as our Q_3 and R_3.

(*d*) Our fourth gross dividend is **1550.** We subtract, $6abc+b^3$ (=1176+64=1240) therefrom, obtain 310 as our

fourth actual working dividend, divide it by 147 and set down 1 and 163 as our Q_4 and R_4.

(e) Our next gross dividend is 1630. We subtract $3ac^2+3b^2c$ (=1029+336=1365) therefrom, get 265 as our fifth actual working dividend, divide it by 147; and so on.

Note:—The divisor should *not* be too small. Its ultra-smallness will give rise to big quotients (sometimes of several digits), the insufficiency of the remainders for the subtractions to be made and other such complications which will confuse the student's mind.

In case the divisor actually happens to be too small, two simple devices are available for surmounting this difficulty.

(i) Take the first four (or 5 or 6) digits as one group and extract the cube root. For example, suppose we have to find out the cube root of 1346, 085. Our chart will then have to be framed thus:

363	: 1, 346 :	086	
	: 1 331 :	15	
	: 11 :		

Let us now take an actual concrete example and apply this method for extracting the cube root of **6334625**:

	: 6334 :	6	2	5	
972	: 6 832 :	502	166	312	1944
	: 18 :	5	0	2	......

(a) $Q_1=18$; $R_1=502$; and Divisor (D)=972.

(b) No subtraction being needed at this point, divide 5026 by 972 and put down 5 and 166 as Q_2 and R_2.

(c) Our third gross dividend (G.D) is 1662; subtract $3ab^2$ (=1350) from 1662, divide the resultant. Actual dividend (AD) i.e. 312 by 972 and set down 0 and 312 as Q_3 and R_3.

(*d*) Our next GD is 3125. "Subtract $6abc+b^3$ (=0+125= 125) from 3125, divide the AD (3000) by 972 and put down 2 and 1944 as Q_4 and R_4 and so on,

Or, Secondly, multiply E (the given expression) by 2^3, 3^3, 43 or 53 etc., (as found necessary and most convenient) find the cube root and then divide the CR by 2, 3, 4, 5 etc. For instance, instead of taking 3 (as the divisor), take 3×4^3 ($=3\times64=192$), find the cube root and divide it by 4.

Here again, a concrete example may be worked out by both the methods:

First Method

(a) $Q_1=1$; $R_1=1$; and D=3

$$\begin{array}{r|l} & 2 : \quad 0 \quad 0 \quad 0 \; 0 \\ 3 : & \quad : 1 \quad 4 \; 1 \; 0 \; 2 \\ \hline & : 1 : \quad 26 \end{array}$$

(b) Now, GD=AD=10. Divided by 3, it gives 2 and 4 as Q_2 and R_2.

(c) The third GD is 40. From this subtract $3ab^2$ (=12). After this subtraction, tho AD is 28. Divide this by 3 and put down 6 and 10 as Q_3 and R_3.

(d) The fourth GD is 100. From this deduct $6abc+b^a$ (=72+8=80). The AD is 20.

Now, as for dividing this 20 by 3, the directly apparent Q_4 and R_4 are 6 and 2. But the *actual* quotient and Remainder are difficultto determine (because of the smallness of the divisor) and the insufficiency of the Remainders for the next subtractions and a good number of trial digits may fail before one can arrive at the correct figures! This is why the other method is to be preferred in such cases. And then the working will be as follows:

Multiplying 2 by b^3, we get 250.

$$\begin{array}{r|l} & 250 : \quad 0 \quad 0 \quad 0 \\ 108 : & \quad : 34 \; 124 \; 196 \; 332 \\ \hline & : \; 6 \; : \; 2 \; 9 \quad 9 \ldots \end{array}$$

(a) Q_1 and $R_1=6$ and 34

(b) $E_2=340$. Dividing this by 108, we have $Q_2=2$ and $R_2=124$.

(c) $3ab^2=72$. Deducting this from 1240, we get 1168. Dividing this by 108, $Q_3=9$ and $R_3=196$

(d) $6abc+b^S=648+8=656$ ∴ The Working Dividend $=1960-656=1304$. Dividing this by 108, we have $Q_4=9$ and $R_4=332$ ∴ The CR=6·299—

∴ Dividing by 5, the actual cube root=1·259—

(2) Let us take another concrete example i.e. 3 12. We multiply 12 by 5^3 and put 1500 down as the Total Dividend. And we take the first four digits as one group.

	1500:	0	0	0	
363 :	:	169	238	400	387
	: 11 :	4	4	7	...

(a) Thus $Q_1=11$ and $R_1=169$

(b) Dividing 1690 by 363, we have $Q_2=4$ and $R_2=238$

(c) $3ab^2=528$ ∴ Working Dividend$=2380-528=1852$
∴ Dividing it by 363, we have $Q_3=4$ and $R=400$

(d) $6abc+b^3=1056+64=1120$. Deducting this from 4000, we get 2928. Dividing this by 363, $Q_4=7$ and $R=387$

∴ The CR=11·447 etc.

∴ The cube root of the original E=2·289...

Some more examples may be taken :

(1) (a) E=1728 ; $Q_1=1$; D=3 ; and $R_1=0$

	1 :	7	2	8
3 :	:	0	1	0
	: 1 :	2 .	0	0

(exact cube)

(b) 7 divided by 3 gives 2 and 1 as Q_2 and R_2

(c) Third Gorss Dividend=12 ; $3ab^a=12$; ∴ Actual dividend=0 ∴ $Q_3=0$ and $R_3=0$

(d) Fourth gross dividend=8 ; $6abc+b^8=0+8=8$
∴ Subtracting the latter from the former, $Q_4=0$ and $R_4=0$

∴ The CR=12

N.B. The obvious second proof speaks for itself.

(2) (a) Here E=13824 ; D=12 ;
$Q_1=2$; and $R_1=5$

```
      13:  8   2 4
12 :     :5  10 6
   : 2:  4·  0 0
```

(Perfect cube)

(b) $\frac{58}{12}$ gives $Q_2=4$; and $R_2=10$

(c) Gross Dividend=102 ; $3ab^2=96$.
∴ Actual dividend=6. Divided by 12, this gives
$Q_3=0$ and $R_3=6$

(d) G.D=64 ; $6abc+b^3=0+64$ ∴ A.D=0 ∴ $Q_4=0$ and $R_4=0$ ∴ The CR is 24.

(3) Here E=33, 076, 161 ; $Q_1=3$; D=27 ; and $R_1=6$

(a) $Q_1=3$; and $R_1=6$

```
27:33: 6067462 1 0601
   3 . 2  1  0  0  0
```

(complete cube).

(b) GD=AD=60 ; Divided by 27, this gives 2 &6 as Q_2 and R_2

(c) GD is now 67 ; $3ab^2=36$ ∴ A.D=31. Divided by 27, this gives us 1 and 4 as Q_3 and R_3.

(d) GD is 46 ; $6abc+b^3=36+8=44$ ∴ AD=2 And, divided by 27, this gives 0 and 2 as Q_4 and R_4.

(e) GD is 21 ; $3ac^2+3b^2c=9+12=21$ ∴ AD=0 ; and, divided by 27, this gives 0 and 0 as Q_5 and R_5.

(f) GD is 6 ; $3bc^2=6$ ∵ AD=0. And, divided by 27, this gives us 0 and 0 as Q_6 and R_6.

(g) GD=1 ; $c^3=1$; AD=0. Divided by 27, this gives us 0 and 0 as Q_7 and R_7 ∴ The CR is 321.

N.B.—The second proof is clearly there before us.

(4) E=101

```
     101 :   0  0   0   0
48 :     : 37 82 148 112 40
   : 4.  :  6  5   9   5
```

(a) $Q_1=4$; $R_1=37$; and D=48

(b) GD=AD=370 ; and, divided by 48, this gives us 6 and 82 as Q_2 and R_2

(*c*) **GD=820 ; $3ab^2=432$ ∴ AD=388.** and, **divided** by 48, this gives us **5** and 148 as Q_3 and R_3

(*d*) **G.D=1480** ; and $6abc+b^3=720+216=936$; ∴ **AD=544.** And this, divided by 48, gives **us** 9 and **112** as Q_4 and R_4

(e) **GD=1120** ; and $3ac^2+3b^2c=300+540=840$ ∴ **AD=280.** And, divided by 48, this gives **us 5** and 40 as Q_5 and R, ; and so on.

(5) **E=29791**

(a) Here $Q_1=3$; and $R_1=2$; and **D=27**

(b) **GD=AD=27** ; and, divided by 27, this gives us **1** and **0** as Q_2 and R_2.

	29:	7	9	1	
27 :		**: 2**	**0**		
	: 3:	**1**	**.0**		

(complete cube)

(c) **GD=9** ; and $3ab^2=9$; ∴ **AD=0,** and, divided by 27, this gives us **0** and **0** as Q, and R_3 ∴ The CR **is** 31.

N.B. :—The proof is there-before us as usual.

(6) The given expression **(E)=83,** 453, **453**

(*a*) $Q_1=4$; $R_1=19$; and **D=48.**

(b) **GD=AD=194.** And, divided by 48, this gives us **3** and 50 as Q_2 and R_2

	: 83 :	4	5	3	4	5	3
48 :	: 19		50	61	82	47	34
	: 4 :	3	7.	**0**	**0**	**0**	(exact cube)

(c) **GD=505** ; and $3ab^a=108$ ∴ **AD=397.** And, divided by 48, this gives us 7 and 61 as Q, and R_3

(d) **GD=613** ; and $6abc+b^3=504+27=531$ ∴ **AD=82** And, divided by 48, **this** gives us **0** and 82 as Q_4 & R_4

(e) **GD=824** ; and $3ac^2+3b^2c=588+189=777$ ∴ AD =47 And this, divided by 48, **gives** us **0** and 47 as Q_5 & R_5

(*f*) $GD-3bc^2=475-441=34.$ ∴ $Q_6=0$ and $R_6=34$

(*g*) **GD=343** ; and $c^3=343$ ∴ **AD=0** ∴ $Q_7=R_7=0$

∴ The CR is 437.

N.B.:—The proof is there as usual.

(7) E=84, 604, 519

(*a*) $Q_1=4$;		84 :	6	0	4	5	1	9	
D=48;	48 :	: 20	62	80	129	80	72		
and $R_1=20$	:	4 :	3	9 .	0	0	0		(perfect cube)

(b) GD=AD=206. And, divided by 48, this gives us 3 and 62 as Q_2 and R_2

(c) GD=620; and $3ab^2=108$ ∴ AD=512. And, divided by 48, this gives us 9 and 80 as Q, and R_3

(d) GD=804; and $6abc+b^3=648+27=675$ ∴ AD=129. And, divided by 48, this gives us 0 and 129 as Q_4 & R_4

(e) GD=1295; and $3ac^2+3b^2c=972+243=1215$ ∴ AD=80. And, divided by 48, this gives us 0 and 80 as Q_5 and R_5

(f) GD=801; and $3bc^2=729$ ∴ AD=72. And, divided by 48, this gives us 0 and 72 as Q_6 and R_6

(*g*) GD=729; and $C^3=729$ ∴ AD=0 ∴ $Q_7=0$ and $R_7=0$

∴ The CR is 439

N.B.:—The proof is there as usual.

(8) E=105, 823, 817

(*a*) $Q_1=4$;		: 105 :	8	2	3	8	1	7	
$R_1=41$;	48 :	: 41	82	90	56	19	2		
and D=48	:	4 :	7	3·	0	0	0		(complete cube)

(*b*) GD=AD=418. And, divided by 48, this gives us 7 and 82 as Q_2 and R_2

(c) GD=822; and $3ab^2=588$ ∴ AD=234; and, divided by 48, this gives us 3 and 90 as Q_3 and R_3

(d) GD=903; and $6abc+b^3=504+343=847$ ∴ AD=56. And, divided by 48, this gives us zero and 56 as Q_4 & R_4

(e) GD=568; and $3ac^2+3b^2c=108+441=549$ ∴ AD =19. And divided by 48, this gives us zero and 19 as Q_5 & R_5

(*f*) GD=191 ; and $3bc^2$=189 ∴ AD=2 ; and, divided by 48, this gives us zero and 2 as Q_6 and R_6

(*g*) GD=27 ; and C^3=27 ∴ AD=0 ∴ O_7=0 and R_7=0

∴ The CR=473

N.B.:—The proof is there as usual.

(9) E=143, 055, 667

```
     143:   0  5   5  6 6 7
75:      : 18 30  20 17 5 2
  :  5 :  2 3 . 0 0  0 (exact cube)
```

(*a*) Q_1=5 ; R_1=18 ; and D=75

(*b*) GD=AD=180 ; and, divided by 75, this gives us 2 and 30 as Q_2 and R_2

(*c*) GD=305 ; and $3ab^a$=60 ∴ AD=245 ; and, divided by 75, this gives us 3 arid 20 as Q_3 and R_3

(*d*) GD=205 ; and $6abc+b^3$=180+8=188 ∴ AD=17. And, divided by 75, this gives us 0 and 17 as Q_4 and R_4

(*e*) GD=176 ; and $3ac^2+3b^2c$=135+36=171 ... AD=5. And, divided by 75, this gives 0 and 5 as Q_5 and R_5

(*f*) GD=56 ; and $3bc^2$=54 ∴ AD=2 ; and, divided by 75, this gives 0 and 2 as Q_6 and R_6.

(*g*) GD=27 ; and c^3=27 ∴ AD=0 ∴ Q_7=0 and R_7=0

∴ The CR is 523

N.B.:—The proof is there as usual.

(10) E=248, 858, 189.

```
     : 248 :  8   5  8   1   8  9
108 :       : 32 112 81 162 155 72
    :    6 :  2   9.. 0  0  0(perfect cube)
```

(a) Q_1=6 ; R_1=32 ; and D=108

(b) GD=AD=328. And, divided by 108, this gives us 2 and 112 as Q_2 and R_2

(c) GD=1125 ; and $3ab^2$=72 ∴ AD—1053 ; and, divided by 108, this gives us 9 and 81 as Q_3 and R_3.

(d) GD=818 ; and $6abc+b^3$=648+8=656 ∴ AD=162. And, divided by 108, this gives 0 and 162 as Q_4 & R_4

(*e*) GD=1621 ; and $3ac^a+3b^2c$=1458+108=1506 ∴ AD=55. And, divided by 108, this gives us 0 and 55 as Q_5 and R_5.

(*f*) GD=558 ; and $3bc^2=486$ ∴ AD=72 ; and, divided by 108, this gives us 0 and 72 and Q_6 and R_6

(g) GD=729 ; and $C^3=729$ ∴ AD=0 ; $Q_7=0$ and $R_7=0$

∴ The CR is 629

N.B. :—The proof is there as usual.

(11) 11, 345, 123, 223

Note:—The cube root in this case being of four digits, the method obtained from the expansion of $(a+b+c)^3$ will naturally not sufficefor this purpose ; and we shall have to expand $(a+b+c+d)^3$ and vary the above procedure in accordance therewith. This is, of course, perfectly reasonable.

The Schedule of *Digits*

The Analytical digit-schedule for $(a+b+c+d)^3$ now stands as follows:

(a) First digit (9 zeros)$=a^3$—
(b) Second digit (8 zeros)$=3a^2b$—
(c) Third digit (7 zeros)$=3ab^2+3a^2c$
(d) Fourth digit (6 zeros)$=6abc+b^3+3a^2d$
(e) Fifth digit (5 zeros)$=6abd+3ac^2+3b^2c$—
(*f*) Sixth digit (4 zeros)$=6acd+3bc^2+3b^2c$—
(g) Seventh digit (3 zeros)$=6bcd+3ad^2+c^3$—
(*h*) Eighth digit (2 zeros)$=3bd^2+3c^2d$—
(*i*) Ninth digit (1 zero)$=3cd^2$—
(*j*) Tenth digit (no zero)$=d^3$

Consequent *Subtractions*

(1) Q_1 and R_1 by mere inspection.

(2) Q_2 and R_2 by simple division (without any subtraction whatsoever).

(3) Prom all the other Gross Dividends, subtract :

(3) $3ab^2$
(4) $6abc+b^a$
(5) $6abd+3ac^2+3b^2c$
(6) $6acd+3bc^2+3b^2d$

(7) $6bcd+3ad^2+c^3$
(8) $3bd^2+3c^2d$
(9) $3cd^2$
(10) d^3

respectively, in order to obtain the actual working dividend and thenoe deduce the required Q and R.

Note:—It will be noted that, just as the equating of d to zero in $(a+b+c+d)^3$ will automatically give us $(a+b+c)^3$ exactly so will the substitution of zero for d in the above schedule give us the necessary schedule for the three-digit cube root.

As we go higher and higher up (with the number of digits in the cube root), the same process will be found at work. In other words, there is a general formula for n terms (n being any positive integer); and all these are only special applications of that formula (with n equal to **2, 3, 4** and so on). This general form of the formula, we shall take up and explain at a later stage in the student's progress.

In the meantime, just now, we explain the application of the (a+b+c+d)3 schedule to the present case.

Application to the Present Case

	11:	3	4	5	1	2	3	2	2	**3**	
12:	:	3	9	22	37	59	76	69	62	34	
: 2	:	2	4	7·	0	0	0	0			(exact cube)

(*a*) $Q_1=2$; $R_1=3$; and Divisor is 12

(*b*) GD=AD=33. Dividing this by 12, we get $Q_2=2$ and $R_2=9$

(*c*) GD=94 ; and $3ab^2=24$. ∴ AD=70 ∴ $Q_3=4$ and $R_3=22$

(*d*) GD=225 ; and $6abc+b^3=96+8=104$; ∴ AD=121 ∴ $Q_4=7$ and $R_4=37$

(*e*) GD=371 ; and $6abd+3ac^2+3b^2c=168+96+48=312$. ∴ AD=59 ∴ $Q_5=0$ and $R_5=59$.

(*f*) GD=592 ; and $6acd+3bc^2+3b^2d=336+96+84=516$ D=76 ∴ $Q_6=0$ and $R_6=76$.

(*g*) GD=763 ; and $6bcd+3ad^2+c^3=336+294+64=694$ ∴ AD=69 ∵ $Q_7=0$ and $R_7=69$

(*h*) GD=692 ; and $3bd^2+3c^2d=294+336=630$ ∴ AD=62. ∴ $Q_8=0$ and $R_8=62$

(*i*) GD=622 ; and $3cd^2=588$ ∴ AD=34 ∴ $Q_9=0$ and $R_9=34$

(*j*) GD=343 ; and $d^3=343$ ∴ AD=0 ∴ $Q_{10}=0$ and $R_{10}=0$ ∴ The cube root is 2247

N.R.:—The ocular proof is there, as usual. This is the usual procedure. There are certain devices, however, which can help us to over-come all such difficulties; and, if and when a simple device is available and can serve our purpose, it is desirable for us to adopt it and minimise the mere mechanical labour involved and not resort to the other ultra-laborious method.

The devices are therefore explained hereunder :

The First Device

The first device is one which we have already made use of, namely, the reckoning up of the first 4, 5 or 6 digits as a ***group*** by itself. Thus, in this particular case :

	:	**11,**	**345 :**	**1**	2	3	2	2	3
1452	:	10	648 :	697	1163	412	363	62	34
	:	**22**	:	**4**	7.	0	0	0 (complete cube)	

(*a*) Q_1 (by the same process) is the double-digit number 22; $R_1=697$; and D=1452

(b) GD=AD=6971 ; and, divided by 1452, this gives us 4 and 1163 as Q_2 and R_2

(c) GD=11632 ; and $3ab^2=1056$ ∴ AD=10576 ; and, divided by 1452, this gives us 7 and 412 as Q_3 and R_3

(*d*) GD=4123 ; and $6abc+b^3=3696+64=3760$ ∴ AD= 363 ; and, divided by 1452, this gives us 0 and 363 as Q_4 and R_4

(e) GD=3632 ; and $3ac^2+3b^2c=3234+336=3570$
$\therefore$ AD=62 ; and, divided by the same divisor 1452, this gives us zero and 62 as Q_5 and R_5

(*f*) GD=622 ; and $3bc^2=588$ $\therefore$ AD=34. So $Q_6=0$ and $R_6=34$

(g) GD=343 ; and $c^3=343$ $\therefore$ AD=0 $\therefore$ $Q_7=0$ and $R_7=0$
$\therefore$ The C.R.=2247

N.B. :—(1) And the Proof is there before us, as usual.
(2) By this device, we avoid the complication caused by shifting from $(a+b+c)^3$ to $(a+b+c+d)^3$. It, however, suffers from the draw-back that we have first to find the double-digit Q, cube it and subtract it from the first five-digit portion of the dividend and that all the four operations are of big numbers.

Second Device

This is one in which we do not magnify the first group of digits but substitute $(c+d)$ for c all through and thus have the same $(a+b+c)^3$ procedure available to us. But, after all, it is only a slight alteration of the first device, whereby, instead of a two-digit quotient-itam at the commencement, wc will be having exactly the same thing at the end.

The real desideratum is a formula which is applicable not only to two-digit, three-digit, or four-digit cube roots but one which will be automatically and universally applicable. But this we shall go into at a later stage of the student's progress.

In the meantime, a few more illustrative instances are given hereunder (for further elucidation of—or at least, the student's practice in, the methods hereinabove explained :

(1) E=12, 278, 428, 443.

Here too we may follow the full procedure or first ascertain the first two-digit portion of the cube root of 12, 278, treat the whole five-digit group as one packet and extract the cube root of the whole given expression in the usual way. The procedure will then be as follows:

(i) *Single-digit* method

	12:	2 7	8	4	2	8	4	4	3	
12:	: 4	6	13	27	22	33	44	3	34	
:	2:	3 0	7 · 0	0	0 0					(perfect cube)

(*a*) $Q_1=2$; $R_1=4$; and $D=12$

(*b*) $GD=AD=42$ $\therefore$ $Q_2=3$ and $R_2=6$

(*c*) $GD=67$; and $3ab^2=54$; $\therefore$ $AD=13$ $\therefore$ $Q_3=0$ and $R_3=13$

(d) $GD=138$; and $6abc+b^3=0+27=27$ $\therefore$ $AD=111$ $\therefore$ $Q_4=7$ and $R_4=27$

(*e*) $GD=274$; and $6abd+3ac^2+3b^2c=252+0+0=252$; $\therefore$ $AD=22$ $\therefore$ $Q_5=0$ and $R_5=22$

(*f*) $GD=222$; and $6acd+3bc^2+3b^2d=0+0+189$ $\therefore$ $AD=33$ $\therefore$ $Q_6=0$ and $R_6=33$

(*g*) $GD=338$; and $6bcd+3ad^2+c^3=0+294+0=294$ $\therefore$ $AD=44$ $\therefore$ $Q_7=0$ arid $R_7=44$

(*h*) $GD=444$; and $3bd^2+3c^2d=441+0=441$ $\therefore$ $AD=3$ $\therefore$ $Q_8=0$ and $R_8=3$

(i) $GD=34$; and $3cd^2=0$ $\therefore$ $AD=34$ $\therefore$ $Q_9=0$ and $R_9=34$

(*j*) $GD=343$; and $d^3=343$ $\therefore$ $AD=A0$, $Q_{10}=0$ and $R_{10}=0$

$\therefore$ The CR=2307

N.B. :—The proof is before us, as usual.

(i) Two-Digit method

Preliminary Work

	: 12 :	278
12:	: 4	
	: 2 :	3...

$\therefore$ Q_1 (of two digits) is 23;

	12278 :	4	2	8	4
1587 :	12167 :	111	1111	33	338
	: 23 :	0	7 .	0	

(a) Q_1 (of two digits)=23 ; R_1=111 ; and D=1587

(b) GD=AD=1114 ∴ Q_2=0 ; and R_2=1114

(c) GD=11142 ; and $3ab^2=0$ ∴ AD=11142 ∴ Q_3=7 & R_3=33

(d) GD=338 ; and $6abc+b^3=0$ ∴ AD=338 ∴ Q_4=0 & R=338

(*e*) GD=3384 ; and $3ac^a+3b^2c$=3381+ 0=3381 ∴ AD=3 ∴ Q_5=0 and R_5=3

(f) GD=34 ; and $3bc^2=0$ ∴ AD=34 ∴ Q_6=0 & R_6=34

(*g*) GD=343 ; and d^3=343 ∴ AD=0, Q_7=0 and R_7=0

∴ The CR is 2307

N.B. :—The proof is before us, as usual.

(2) E=76, 928, 302, 277.

(i) Single-digit method

	: 76 :		9	2	8	3	0	2	2	7	7
48:		:	12	33	44	56	59	44	29	13	2
	: 4 :		2	5	3.	0	0	0	0	(Exact cube)	

(a) Q_1=4 ; R_1=12 ; and D=48

(b) GD=AD=129 ∴ Q_2=2 and R_2=33

(c) GD=332 ; and $3ab^2$=48 ∴ AD=284 ... Q_3=5 & R_3=44

(*d*) GD=448 ; & $6abc+b^3$=240+8=248 ∴ AD=200 ∴ Q_4=3 and R_4=56

(e) GD=563 ; & $6abd+3ac^2+3b^2c$=144+300+60=504 ∴ AD=59 ∴ Q_5=0 and R_5=59

(*f*) GD=590 ; & $6acd+3bc^2+3b^2d$=360+150+36=546 ∴ AD=44 ; ∴ Q_6=0 and R_6=44

(g) GD=442 ; and $6bcd+3ad^2+c^3$=180+108+125=413 ∴ AD=29 ; ∴ Q_7=0 and R_7=29

(h) GD=292 ; and $3bd^2+3c^2d$=54+225=279 ∴ AD=13 ∴ Q_8=0 and R_8=13

(i) **GD=137**; and $3cd^2=135$ ∴ **AD=2** ∴ $Q_9=0$ and $R_9=2$

(j) **GD=27**; and $d^3=27$ ∴ **AD=0** ∴ $Q_{10}=0$ and $R_{10}=0$ ∴ The **CR=4253**

N.B. :—Proof as usual.

(ii) Double-digit ***method***

or, secondly,

∴ Q, (double-digit)=&

	: 76 :		9
48 :		: 12	33
	: 4 :		2

	: 769 28 :	3	0	2	2	7	7
5292 :	740 88 :	2840	1943	404	137	13	2
	: 42 :	5	3	0.	0	0	

(a) $Q_1=42$; **R=2840**; and **D=5292**

(b) **GD=AD=28403** ∴ $Q_2=5$; and $R_2=1943$

(c) **GD=19430**; and $3ab^2=3150$ ∴ AD=16280 ∴ $Q_3=3$ and $R_3=404$

(d) **GD=4042**; and $6abc+b^3=3780+125=3905$ ∴ **AD=137** ∴ $Q_4=0$ and $R_4=137$

(e) **GD=1372**; and $3ac^2+3b^2c=1134+225=1359$ ∴ **AD=13** ∴ $Q_5=0$ and $R_5=13$

(f) **GD=137**; and $3bc^2=135$ ∴ **AD=2** ∴ $Q_6=0$ and $R_6=2$

(g) **GD=27**; and $c^3=27$ ∴ **AD=0**; $Q_7=0$ and $R_7=0$

∴ **The** CR is 4263

N.B. Exactly as above.

(3) **E=355,** 045, 312, 441

(i) ***Single-Digit*** method.

	355:	0	4	6	3	1	2	4	4	1
147 :	:	12	120	28	138	39	55	19	6	0
	: 7 :	0	8	1	·0					

(a) $Q_1=7$; $R_1=12$; and **D=147**

(b) **GD=AD=120**; ∴ $Q_2=0$ and $R_2=120$

(c) GD=1204 ; and $3ab^S=0$ ∴ AD=1204 ∴ $Q_3=8$ & $R_3=28$

(d) GD=285 ; and $6abc+b^8=0$ ∴ AD=285 ; $Q_4=1$ and $R_4=138$

(e) GD=1383 and $6abd+3ac^S+3b^Zc=0+1344+0=1344$ ∴ AD=39 ∴ $Q_5=0$ and $R_5=39$

(f) GD=391 ; and $6acd+3bc^2+3b^Sd=336+O+0$ ∴ AD=55 ∴ $Q_6=0$ and $R_6=55$

(g) GD=552 ; and $6bcd+3ad^Z+c^S=0+21+512=533$ ∴ AD=19 ∴ $Q_7=0$ and $R_7=19$

(h) GD=194 ; and $3bd^2+3c^2d=0+192$ ∴ AD=2 ∴ $Q_8=0$ and $R_8=2$

(i) GD=24 ; and $3cd^2=24$ ∴ AD=0 ; Q=0 and R=O

(j) GD=1 ; and $d^3=1$ ∴ AD=0, Q=O and R=O

∴ The CR is 7081

N.B. As above.

(ii) Double-digit method.

	:	355045	:	3		1	2	4	4	1
14700	:	343000	:	12045	2853	391		40	2	0
	:	70	:	8		1 · 0		0		0

(a) Q_1 (of 2 digits)=70 ; $R_1=12045$; and D=14700

(b) GD=AD=120453 $Q_2=8$ and $R_2=2853$

(c) GD=28531 ; and $3ab^a=13440$ ∴ AD=15091 ∴ $Q_3=1$ and $R_3=391$

(d) GD=3912; & $6abc+b^a=3360+512=3872$ ∴ AD=40 ∴ $Q_4=0$ and $R_4=40$

(e) GD=404 ; & $3ac^2+3b^2c=210+192=402$ ∴ AD=2 ∵ $Q_5=0$ and $R_5=2$

(f) GD=24 ; and $3bc^2=24$ ∴ AD=0 ∴ $Q_6=0$ and $R_6=0$

(g) GD=1 ; and $d^3=1$ ∴ AD=0 ∴ $Q_7=0$ and $R_7=0$

∴ The CR is 7081

N.B. As above.

(4) E=792, 994, 249, 216

(i) Single-Digit method.

	792:	9	9	4	2	4	9	2	1	6
243 :	:	63	153	216	158	199	152	72	56	21
	: 9 :	2		5	6 .	0	0	0	0	

(a) $Q_1=9$; $R_1=63$; and D=243

(b) GD=AD=639 $\therefore$ $Q_2=2$; and $R_2=153$

(c) GD=1539 ; and $3ab^2=108$ $\therefore$ AD=1431 $\therefore$ $Q_3=5$ and $R_3=216$

(d) GD=2164 ; & $6abc+b^3=540+8=548$ $\therefore$ AD=1616 $\therefore$ $Q_4=6$ and $R_4=158$

(e) GD=1582 ; and $6abd+3ac^2+3b^2c=648+675+60=1363$ $\therefore$ AD=199 $\therefore$ $Q_5=0$ and $R_5=199$

(f) GD=1994 ; and $6acd+3bc^2+3b^2d=1620+150+72=1842$ $\therefore$ AD=152 $\therefore$ $Q_6=0$ and $R_6=152$

(g) GD=1529; & $6bcd+3ad^2+c^3=360+972+125=1427$ $\therefore$ AD=72 $\therefore$ $Q_7=0$ and $R_7=72$.

(h) GD=722 ; & $3bd^2+3c^2d=216+450=666$ $\therefore$ AD=56 $\therefore$ $Q_8=0$ and $R_8=56$

(i) GD=561 ; and $cd^2=540$ $\therefore$ AD=21 $\therefore$ $Q_9=0$ and $R_9=21$

(j) GD=216 ; and $d^3=216$ $\therefore$ AD=0 $\therefore$ $Q_{10}=0$ and $R_{10}=0$

$\therefore$ The *CR=9256*

N.B. As above.

Double-Digit *Method*

	792 :	9	
243 :	:	63 153	:
	: 9 :	2	:

$\therefore$ Q_1 (of two digits is 92 and $92^3=778688$. And D=25392.

	792994 :	2	4	9	2	1	6
25392 :	778688 :	14306	16102	1772	1044	56	21
	: 92 :	5	6 .	0	0	0	0 0

(a) $Q_1=92$; and $R_1=14306$

(*b*) GD=AD=143062 ∴ $Q_2=5$; and $R_2=16102$

(c) GD=161024; and $3ab^2=6900$ ∴ AD=154124
∴ $Q_3=6$; and $R_3=1772$

(*d*) GD=17729; and $6abc+b^3=16560+125=16685$
∴ AD=1044 ∴ $Q_4=0$ and $R_4=1044$

(e) GD=10442; and $3ac^2+3b^2c=9936+450=10386$
∴ AD=56 ∴ $Q_5=0$ and $R_5=56$

(*f*) GD=561; and $3bc^2=540$.∴AD=21.∴$Q_6=0$; & $R_6=21$

(g) GD=216; and $D^3=216$ ∴ AD=0 ∴ $Q_7=0$ and $R_7=0$

∴ The CR is 9256

N.B. As above.

Note:—It must be admitted that, although the double-digit method uses the $(a+b+c)^3$ schedule and avoids the $(a+b+c+d)^3$ one, yet it necessitates the division, multiplication and subtraction of big numbers and in therefore likely to cause more mistakes. It is obvious-by better and safer to use the $(a+b+c+d)^3$ and deal with smaller numbers.

In this particular case, however, as the given number terminates in an even number and is manifestly divisible by 8 (and perhaps 64 or even 512), we can (in this case) utilise a third method which has already been explained (in the immediately preceding chapter), namely, divide out by 8 (and its powers) and thus diminish the magnitude of the given number. We now briefly remind the student of that method.

Third Method

8 :	7 9 2	99 4	249	216
	8 9 9	124	281	152
	8: 12	390	535	144
	1,	548,	816,	893

	: 1 :		5	4	8	8	1	6	8	9	3
3 :	:	0	2	6	16	36	55	74	76	34	
	: 1 :		1	5	7	0	0	0	0		

(a) $Q_1=1$ and $R_1=0$

(b) GD=AD=05 $\therefore$ $Q_2=1$ and $R_2=2$

(*c*) GD=24 ; and $3ab^2=3$ $\therefore$ AD=21 $\therefore$ $Q_3=5$ and $R_3=6$

(d) GD=68 ; and $6abc+b^3=30+1=31$ $\therefore$ AD=37 $\therefore$ $Q_4=7$; and $R_4=16$

(e) GD=168 ; and $6abd+3ac^2+3b^2c=42+75+15=132$ $\therefore$ AD=36 $\therefore$ $Q_5=0$ and $R_5=36$

(f) GD=361 ; and $6acd+3bc^S+3b^Sd=210+76+21=306$ $\therefore$ AD=55 $\therefore$ $Q_6=0$ and $R_6=55$

(*g*) GD=556 ; and $6bcd+3ad^2+c^3=210+147+125=482$ $\therefore$ AD=74 $\therefore$ $Q_7=0$ and $R_7=74$

(h) GD=748; and $3bd^2+3c^2d=147+525=672$ $\therefore$ AD=76 $\therefore$ $Q_8=0$ and $R_8=76$

(*i*) GD=769 ; and $3cd^2=735$ $\therefore$ AD=34 $\therefore$ $Q_9=0$; and $R_9=34$

(j) GD=343 ; and $d^3=343$ $\therefore$ AD=0 $\therefore$ $Q_{10}=0$; and $R_{10}=0$

$\therefore$ The C.R. (of the sub-multiple)=1157

$\therefore$ The C.R. of the given number=9256

Or, Fourthly, the derived submultiple may be dealt with (by the two-digit method) thus :—

	1548:	8	1	6	8	9	3
363 :	1331 :	217	363	266	221	76	34
:	1 1 :	5	7	0	0	0	

(a) $Q_1=11$; $R_1=217$; and D=363

(b) GD=AD=2178 $\therefore$ $Q_2=5$ and $R_2=363$

(c) GD=3631 ; and $3ab^S=825$ $\therefore$ AD=2806 $\therefore$ $Q_3=7$ & $R_3=265$

(d) GD=2656 ; and $6abc+b^3=2310+125=2435$ $\therefore$ AD=221 $\therefore$ $Q_4=0$ and $R_4=221$

(*e*) GD=2218 ; and $3ac^2+3b^2c=1617+525=2142$ $\therefore$ AD=76 $\therefore$ $Q_5=0$ and $R_5=76$

(*f*) GD=769; and $3bc^2$=735 ∴ AD=34 ∴ Q_6=0; & R_6=34

(g) GD=343; and c^3=343 ∴ AD=0 ∴ Q_7=0 and R_7=0

∴ The cube root of the sub-multiple is 1157

∴ The CR of the original number=*9256*

N.B. As above.

(5) E= : 2, 840, 363, 499, 528
8 :

355, 045, 312, 441,

	355	:	0		4	
147 :		:	12	120		
:	7	:	0	8		1

This very number having already bean dealt with (in example 3 of this very series, in this very portion of this subject), we need not work it all out again. Suffice it to say that, because. 7081 is the cube root of this derived sub-multiple,

∴ The C.R. of the original number is 14162

Note:—All these methods, however, fall in one way or another, short of the Vedic ideal of ease and simplicity. And the general formula which is simultaneously applicable to all cases and free from all flaws is yet ahead. But these matters we shall go into, later.

CHAPTER XXXVII

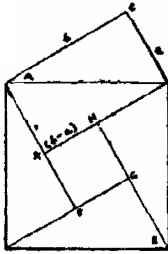
PYTHAGORAX' THEOREM ETC.

Modern Historical Research has revealed—and all the modern historians of mathematics have placed on record the historical fact that the so-called "Pythogoras' Theorem" was known to the ancient Indians long long before the time of Pythagoras and that, just as although the Arabs introduced the Indian system of numerals into the Western world and distinctly spoke of them as the *"Hindu"* numerals, yet, the European importers thereof undiscerningly dubbed them as the Arabic numerals and they are still described everywhere under that designation, similarly exactly it has happened that, although Pythagoras introduced his theorem to the Western mathematical and scientific world long long afterwards, yet that Theorem continues to be known as Pythagoras' Theorem!

This theorem is constantly in requisition in a vast lot of practical mathematical work and is acknowledged by all as practically the real foundational pre-requisite for Higher Geometry (including Solid Geometry), Trigonometry (both plane arid Spherical), Analytical Conics, Calculus (Differential and Integral) and various other branches of mathematics (Pure and Applied). Yet, the proof of such a basically important arid fundamental theorem (as presented, straight from the earliest sources known to the scientific world, by Euclid etc., and as still rxpounded by the most eminent modern geometricians all the world over) is ultranotorious for its tedious length, its clumsy cumbrousness etc., and for the time and toil entailed on it!

There are several Vedic proofs, everyone of which is much simpler than Euclids' etc. A few of them are shown below:

First Proof



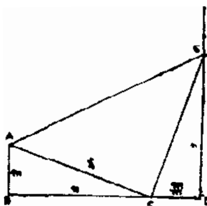
Here, the square AE=the square KG the four congruent right-angled traingles all around it.

Their areas are c^2, $(b-a)^2$ and $4\times\frac{1}{2}ab$ respectively.

$\therefore C^2=a^2-2ab+b^2+4(\frac{1}{2}ab)=a^2+b^2$ Q.E.D.

Second Proof

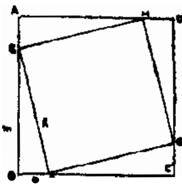
CONSTRUCTION:



CD=AB=m; and DE=BC=n. ∴ ABC and CDE are Congruent; and ACE is right-angled Isosceles. Now, the trapezium ABDE=ABC+CDE+ACE $\therefore \frac{1}{2}mn+\frac{1}{2}h^2+\frac{1}{2}mn=\frac{1}{2}(m+n)\times(m+n)=\frac{1}{2}m^2+mn+\frac{1}{2}n^2$ $\therefore \frac{1}{2}h^2=\frac{1}{2}m^2+\frac{1}{2}n^2 \therefore h^2=m^2+n^2$ Q.E.D.

(*N.B.* Here we have utilised the proposition that the area of a trapezium=+ the altitude×the sum of the parallel sides).

Third Proof

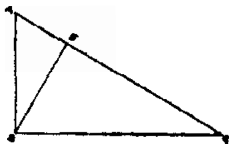


Here, AE=BF=CG=DH=m and EB=FC=GD=HA= n.

Now, the squae AC=the square EG+ the 4 congruent right-angled triangles around it $\therefore h^2+4(\frac{1}{2}mn)=(m+n)^2=m^2+2mn+n^2$ $\therefore h^2=m^2+n^2$ Q.E.D.

Fourth Proof

(The proposition to be used here is that the areas of similar triangles are proportional to the squares on the homologous sides). Here, BD is ⊥ to AC



∴ The triangles ABC, ABD and BCD are similar.

∴ As between (1) the first two triangles and (2) the first and third ones,

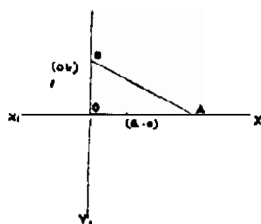
$$\frac{AB^2}{AC^a}=\frac{ADB}{ABC};\ \text{and}\ \frac{BC^2}{AC^a}=\frac{BCD}{ABC}$$

$\therefore$ By addition, $\frac{AB^2+BC^2}{AC^2}=\frac{ADB+BCD}{ABC}=\frac{ABC}{ABC}=1$

$\therefore AB^2+BC^2=AC^2$ Q.E.D.

Fifth Proof

(This proof is from Co-ordinate Geometry. And, as modern Conics and Co-ordinate Geometry (and even Trigonometry) take their genesis from Pythagoras' Theorem, this process would be objectionable to the modern mathematician. But, as the Vedic Sūtras establish their Conics and Co-ordinate Geometry (and even their Calculus), at a very early stage, on the basis of first principles and not from Pythagoras' Theorem (sir), no such objection can hold good in this case.



The proposition is the one which gives us the distance between two points whose co-ordinates have been given. Let the points be A and B and let their co-ordinates be (a, 0) and (0, b) respectively.

Then, RA $\sqrt{(a-0)^2+(0-b)^2}=\sqrt{a^2+b^2}$ $\therefore BA^2=a^2+b^2$ Q.E.D.

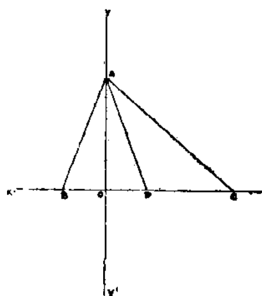
Note:—The Apollonius' Theorem, Ptolemy's Theorem and a vast lot of other Theorems are similarly easy to solve with the aid of the Vedic Sūtras. We shall not, however, go into an elaborate description thereof (except of the Apollonius Theorem) just now but shall reserve them for a higher stage in the student's studies.

CHAPTER XXXVIII

APOLLONIUS' THEOREM

Apollonius' Theorem (sic) is practically a direct and elementary corollary or offshoot from Pythagoras' Theorem. But, unfortunately, its proof too has been beset with the usual flaw of irksome and needless length and laboriousness.

The usual proof is well-known and need not be reiterated here. We need only point out the Vedic method and leave it to the discerning reader to do all the contrasting for himself. And, after all, that is the best way. Isn't it. ?



Well, in any triangle ABC, if D be the mid-point of BC, then $AB^2+AC^2=2(AD^2+BD^2)$. This is the proposition which goes by the name of Apollonius' Theorem and has now to be proved by us by a far simpler and easier method than the one employed by him.

Let AO be the perpendicular from A on BC; let XOX' and YOY' be the axes of co-ordinates; and let BO, OD and OA be m, n and p respectively

$\therefore$ $DB=DC=m+n$

$\therefore$ $AB^2+AC^2=(p^2+m^2)+(m^2+4mn+4n^2+p^2)=$
$2p^2+2m^2+4mn+4n^2$

and $2(AD^2+BD^2)=2[(p^2+n^2)+(m^2+2mn+n^2)]=$
$=2p^2+2m^2+4mn+4n^2$

$\therefore$ $AB^2+AC^2=2(AD^2+DB^2)$ Q.E.D.

Note :—We faintly remember to have read a proof of Apollonius' Theorem on these lines in some publication of Prof. S. L. Loney; but we are not sure. However that may be, this proof (by means of Co-ordinate Geometry) was well-known to the ancient Indian mathematicians and specifically finds its place in the Vedic Sūtras.

And all the Geometrical Theorems about the concurrency of certain straight lines and about the orthocentre,

the circum-centre, the in-centre, the ex-centres, the centroid, the Nine-Points-circle etc., can all be similarly proved (very simply and very easily) by means of Co-ordinate Geometry.

We shall go into details of these theorems and. their Vedic proofs later on; but just now we would just merely point out that, like the "Arabic numerals" and "Pythogoras' Theorem". the "Cartesian" co-ordinates are a historical MISNOMER, no more, and no less.

CHAPTER XXXIX

ANALYTICAL CONICS

Analytical Conics is a very important branch of mathematical study and has a direct bearing on practical work in various branches of mathematics. It is in the fitness of things, therefore, that Analytical Conics should find an important and predominating position for itself in the Vedic system of mathematics (as it actually does).

A few instances (relating to certain very necessary and very important points connected with Analytical Conics) are therefore given here under (merely by way, let it be remembered, of illustration).

I. Equation to the Straight Line,

For finding the Equation of the straight line passing through two points (whose co-ordinates are given.

Say, (9, 17) and (7,-2).

The current method tells us to work as follows:

Take the general equation $y=mx+c$.

Substituting the above values therein,

We have: $9m+c=17$; and $7m+c=-2$.

Solving this simultaneous equation in m and c, we have;

$$9m+c=17$$
$$7m+c=-2$$

$$\therefore 2m=19 \quad \therefore m=9\tfrac{1}{2}$$

Substituting this value of m (in either of the above two equations) we have, $66\tfrac{1}{2}+c=-2 \quad \therefore c=-68\tfrac{1}{2}$. Substituting these values of m and c in the Original General Equation $(y=mx+c)$, we get $y=9\tfrac{1}{2}$ X $-68)$. $\therefore$ Removing fractions, we have $2y=19x-137$. And then, by transposition, we say, $19x-2y=137$. But this method is decidedly too long and cumbrous (and especially for such a petty matter)!

And the *Second Current Method* (which uses the formula $y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$ is equally cumbrous and confusing. It ultimately amounts to the right thing; but it does not make it *clear* and requires **several** more steps of .working!

But the Vedic at-sight, om-line, ***mental method*** (by the ***Parāvartya*** Sutra) enables us to write the answer mechanically down by a ***mere casual look*** at the given co-ordinates. And it is as follows:

The General Equation to the straight line (in its **final** form) is **..x—..y=..** [where the co-efficients of x and y (on the left hand side) and the independent (on the Right hand-side) have to be filled in]. The Siitra tells us to do this very simply by :

(i) putting the difference of the y—co-ordinates as the **x-coefficient** and vice versa; and

(ii) evaluating the independent term on that basis.

For example, in the above example, the **co-ordinates** are: (9, 17) and (7, —2).

(i) so our **x-coefficient** is **17—(—2)=19**

(ii) and our **y—coefficient** is **9—7=2.**

Thus we have 19x**—2y as our** L.H.S. straightaway.

(iii) As for the **absolute** term on the R.H.S., as the straight line in question passes through the two given points, the substitution of the original co-ordinates of **each** of the points must give us the independent term.

So, the substitution of the values **9** and 17 in the L.H.S. of the equation gives us **19×9—2×17=171—34=137 !**

Or Substituting the values 7 and —2 therein, we get 19X7 **—2x—2=133+4=137 !** And that is additional **confirmation** and verification!

But this is not all. There is also a third method by which we can obtain the independent term (on the R.H.S).

And this is with the help of the rule about ***Adyam Antyam and*** Madhyam *i.e.* bc—ad (i.e. the product of the means minus the product of the extremes)! So, we have $17 \times 7 - 9 \times -2 = 119 + 18 = 137$! And this is still further additional confirmation and verification!

So, the equation is :—$19x - 2y = 137$ which is exactly the same as the one obtained by the elaborate current method (with its simultaneous equations transpositions and substitutions etc; galore)! And all the work involved in the Vedic method has been purely ***mental, short,*** simple and easy **!**

A few more instances are given below:

(1) Points (9, 7) and (−7, 2)

∴ The Equation to the straight line joining them is: $5x - 16y = -67$

(2) (10, 5) and (18, 9) ∴ $x = 2y$ (by ***Vilokan*** too)

(3) (10, 8) and (9, 7) ∴ $x - y = 2$ (by ***Vilokan*** too)

(4) (4, 7) and (3, 5) ∴ $2x - y = 1$

(5) (9, 7) and (5, 2) ∴ $5x - 4y = 17$

(6) (9, 7) and (4, −6) ∴ $13x - 5y = 82$

(7) (17, 9) and (13, −8) ∴ $17x - 4y = 253$

(8) (15, 16) and (9, −3) ∴ $19x - 6y = 189$

(9) (a, b) and (c, d)

∴ $x(b - d) - y\,(a - c) = bc - ad$

II. The Gewal Equation and **Two** Straight Lines.

The question frequently arises:—When does the General Equation to a straight line represent two straight lines?

Say, $12x^2 + 7xy - 10y^2 + 13x + 45y - 35 = 0$.

Expounding the current conventional method, Prof. S. L. Loney (the world-reputed present-day authority on the subject) devotes about 15 lines (not of argument or of explanation but of hard solid working) in section 119, example 1 on page 97 of his "Elements of Co-ordinate Geometry", to his model solution of this problem as follows:

Here $a = 12$, $h = \frac{7}{2}$, $b = -10$, $g = \frac{13}{2}$, $f = \frac{45}{2}$ and $c = -35$

$\therefore$ $abc+2fgh-af^2-bg^2-ch^2$

$$=12\,(x-10)\,(x-35)+2\times\frac{45}{2}\times\frac{13}{2}\times\frac{7}{2}-12\left(\frac{45}{2}\right)^2$$

$$-(-10)\left(\frac{13}{2}\right)^2-(-35)\left(\frac{7}{2}\right)^2$$

$$=4200+\frac{4095}{4}-6075+\frac{1690}{4}+\frac{1715}{4}=-1875+\frac{7500}{4}=0$$

$\therefore$ The equation represents two straight lines.

Solving it for x, we have:

$$x^2+x\,\frac{7y+13}{12}+\left(\frac{2y+13}{24}\right)^2=\frac{10y^2-45y+35}{12}+\left(\frac{7y+13}{24}\right)^2$$

$$=\left(\frac{23y-43}{24}\right)^2$$

$$\therefore\ x+\frac{7y+13}{24}=\mp\frac{23y-43}{24}$$

$$\therefore\ x=\frac{2y-7}{3}\ \text{or}\ \frac{-5y+5}{4}$$

$\therefore$ The two straight lines are $3x=2y-7$ and $4x=-5y+5$

Note:—The only comment possible for us to make hereon is that the very magnitude of the numbers involved in the fractions, their multiplications. subtractions etc., *ad infinitum* is appalling and panic-striking and that it is such asinine burden-bearing labour that is responsible for, not as a justification for, but, at any rate, an extenuation for the inveterate hatred which many youngsters and youngstresses develop for mathematics as such and for their mathematics-teachers as such!

We make no reflection on Prof. Loney. He is perhaps one of the best, the finest and the most painstaking of mathematicians and is very highly esteemed by us as such and for his beautiful publications (which arc standard authorities on the various subjects which they deal with). It is the system that we are blaming, (or, at any rate, comparing and contrasting with the Vedic system).

Now, the Vedic method herein is one by which we can immediately apply the "*Ūrdhwa*" Sūtra the *Ādyam Ādyena*

Sūtra and the ***Lopana Sthāpana Sūtra*** and by merely looking at the frightful looking (but really harmless) Quadratic before us, readily by mere *mental* arithmetic, write down the answer to this question and say:—"Yes ; and the straight lines are $3x-2y+7=0$ and $4x+5y-5=0$. How exactly we do this (by mental arithmetic), we proceed to explain presently.

The Vedic ***Method***

(1) By the "*Ūrdhva Tiryak*", the "***Lopana Sthāpana***" and the"Adyam ***Adyena***" Siitras (as explained in some of the earliest chapters), we have (mentally): $12x^2-7xy+10y^2=(3x-2y)$ (&+ti~) and we find 7 and—6 to be the absolute **terms** of the two factors. We thus get $(3x-2y+7)=0$ and $(4x+5y-5)=0$ **as** the two straight lines **represented** by the given **equation.** And that is all there is to it.

$$\begin{array}{r} 3x-2y+7 \\ 4x+5y-5 \\ 12x^2+7xy-10y^2 \\ +13x+45y-35 \\ \hline \end{array}$$

The Hyperbolas and the ***Asymptotes.***

Dealing with the same **principle** and adopting the same procedure in connection with the Hyperbola, the Conjugate Hyperbola and the Asymptotes, in articles 324 and 325 on pages 293 & 294 of his "Elements of Co-ordinate Geometry", Prof. 8. L. Loney devotes **27+14 (=41)** lines in all to the problem and concludes by saying:

"As $3x^2-5xy-2y^2+5x+11y-8=0$ is the equation of the Asymptotes,

$$\therefore 3(-2)c+2,\frac{5}{2}\cdot\frac{11}{2}\frac{(-5)}{2}-3\frac{(11)^2}{2}-(-2)\frac{(5)^2}{2}-c\frac{(-5)^2}{2}=0$$

$$\therefore c=-12$$

∴ **The** Equation to the Asymptotes is $3x^2-5xy-2y^2+5x+11y-12=0$

And consequently the Equation to the Conjugate Hyperbola is $3x^2-5xy-2y^2+5x+11y-16$.

Well; all **this is** not so terrific-looking, because of the very simple fact that all the **working** (according to Art, 116 on pages 95 etc.,) has been taken **for** granted and done "out of Court"

or in private, so to speak. But even then the substitution of the values of a, b, c, **f,** g, and h in the Discriminant to the **General** Equation and so on is, from the Vedic standpoint, **wholly** supererogatory toil and therefore to be avoided.

By the Vedic method, however, we use the same ***Lopan Sthāpana*** the ***Ūrdhva Tiryak*** and the ***Ādyam Ādyena Sūtras ;*** we first get-mentally $3x+y$ and $x-2y$ and then -4 and 3 as the only possibilities in the case; and as this gives us -12 in the product,

$$\begin{array}{r} 3x+y-4 \\ x-2y+3 \\ \hline 3x^2-5xy-2y^2+5x+11y-12 \\ \hline \end{array}$$

we get this **product=0** as the Equation to the Asymptotes; and, as the Conjugate Hyperbola is at the same distance-in the opposite direction from the Asymptotes, we put down the same equation (with only -16 instead of—8) as the required Equation to the Conjugate Hyperbola (and have not got to bother about the complexities of the Discriminants, the inevitable substitutions and all the rest of it)! And that is all.

A few more illustrative instances will not be out of place:

(1) $\left.\begin{array}{l} 8x^2+10xy-3y^2-2x+4y-2=0 \\ \therefore (2x+3y)(4x-y)-2x+4y-2=0 \end{array}\right\}$

$$\begin{array}{r} \therefore 4x-y+1 \\ 2x+3y-1 \\ \hline 8x^2+10xy-3y^2-2x+4y-1=0 \end{array}$$

$\therefore$ The Equation to the Asymptotes is $8x^2+10xy-3y^2-2x+4y=1$; and the Equation to the Conjugate Hyperbola is $8x^2+10xy-3y^2-2x+4y=0$

(2) $y^2-xy-2x^2-5y+x-6=0 \therefore \left.\begin{array}{l} y+x-2 \\ y-2x-3 \end{array}\right\}$

$$\begin{array}{r} \hline y^2-xy-2x^2+x-5y+6=0 \\ \hline \end{array}$$

$\left.\begin{array}{l} \therefore \text{The Asymptotes are } (y+x-2)\ (y-2x-3)=0 \\ \text{And the Conjugate Hyperbola is } y^2-xy-2x^2+x-5y+18=0 \end{array}\right\}$

(3) $55x^2-120xy+20y^2+64x-48y=0 \therefore \begin{array}{l} 11x-2y+4 \\ 5x-10y+4 \end{array}$

$$\begin{array}{r} \hline \therefore 55x^2-120xy+20y^2+64x-48y+16=0 \end{array}$$

∴ This is the Equation to the Asymptotes; and the Equation to the Conjugate Hyperbola is $55x^2-120xy+20y^2+64x-48y+32=0$

(4) $12x^2-23xy+10y^2-25x+26y-14$ ∴ $\begin{matrix} 4x-5y-3 \\ 3x-2y-4 \end{matrix}$

∴ The Asymptotes are: $12x^2-23xy+10y^2-25x+26y+12=0$

And the Conjugate Hyperbola is $12x^2-23xy+10y^2-25x+26y+38=0$

(5) $6x^2-5xy-6y^2+14x+5y+4$ ∴ $\begin{matrix} 2x-3y+4 \\ 3x-2y+1 \end{matrix}$

∴ Independent term$=4$

∴ Two straight lines.

4

CHAPTER XL.

MISCELLANEOUS MATTERS

6 There are also various subjects of a miscellaneous character which are of great practical interest not only to mathematicians and statisticians as such but also to ordinary people in the ordinary course of their various businesses etc., which the modern system of accounting etc., does scant justice to and in which the Vedic Sūtras can be very helpful to them. We do not propose to deal with them now, except to *name* a few of *them* :

(1) Subtractions ;

(2) Mixed additions and subtractions ;

(3) Compound additions and subtractions ;

(4) Additions of Vulgar Fractions etc ;

(6) Comparison of Fractions ;

(6) Simple and compound practice (without taking Aliquot parts etc.)

(7) Decimal operations in all Decimal Work ;

(8) Ratios, Proportions, Percentages, Averages etc. ;

(9) Interest ; Annuities, Discount etc ;

(10) The Centre of Gravity of Hemispheres etc ;

(11) Transformation of Equations ; and

(12) Dynamics, Statics, Hydrostatics, Pneumatics etc., Applied Mechanics etc., etc.

N.B. :—There are some other subjects, however, of an important character which need detailed attention but which (owing to their being more appropriate at a later stage) we do not now propose to deal with but which, at the same time, in view of their practical importance and their absorbingly interesting character, do require a brief description. We deal with them, therefore, briefly hereunder.

Solids, *Trigonometry, Astronomy Etc.*

In Solid Geometry, Plane Trigonometry, Spherical Trigonometry and Astronomy too, there are similarly huge masses of Vedic material calculated to lighten the mathematics students' burden. We shall not, however, go here and now into a detailed disquisition on such matters but shall merely name a few of the important and most interesting headings under which these subjects may be usefully sorted:

(1) The Trigonometrical Functions and their inter-relationships; etc.

(2) Arcs and chords of circles, angles and sines of angles etc;

(3) The converse i.e. sines of angles, the angles themselves, chords and arcs of circles etc;

(4) Determinants and their use in the Theory of Equations, Trigonometry, Conics, Calculus etc;

(5) Solids and why there can be only five regular Polyhedrons; etc., etc.

(6) The Earth's daily Rotation on its own axis and her annual relation around the Sun;

(7) Eclipses;

(8) The Theorem (in Spherical Triangles) relating to the product of the sines of the Alternate Segments i.e. about:

$$\frac{\text{Sin BD}}{\text{Sin DC}} \cdot \frac{\text{Sin CE}}{\text{Sin EA}} \cdot \frac{\text{Sin AF}}{\text{Sin FB}} = 1 \text{ and}$$

(9) The value of 11 (i.e. the ratio of the circumference of a circle to its Diameter).

N.B.:—The last item, however, is one which we would like to explain in slightly greater detail.

Actually, the value of $\frac{\tilde{11}}{10}$ is given in the well-known Anuṣṭub metre and is couched in the Alphabetical Code-Language (described in an earlier chapter):

गोपीभाग्यमधुव्रात-शृङ्गिशोदधिसन्धिग ॥

खलजीवितखाताव गलहालारसंधर ॥

It is so worded that it can bear three different meanings—all of them quite appropriate. The first-is a hymn to the Lord **Śrī Kṛṣṇa**; the second is similarly a hymn in praise of the Lord Shri Shankara; and the third is a valuation of $\frac{\pi}{10}$ to 32 places of Decimals! (with a "Self-contained master-key" for extending the evaluation to ***any*** number of decimal places!

As the student (and especially the non-Sanskrit knowing student) is not likely to be interested in and will find great difficulty in understanding the puns and other literary beauties of the verse in respect of the first two meanings but will naturally feel interested in and can easily follow the third meaning, we give only that third one here:

$$\frac{\pi}{10} = .31415926535897932\ldots$$

on which, on understanding it, Dr. V. P. Dalal (of the Heidelburg University, Germany) felt impelled—as a mathematician and physicist and also as a Sanskrit scholar—to put on record his comment as follows:

> "It shows how deeply the ancient Indian mathematicians penetrated, in the subtlety of their calculations, even when the Greeks had no numerals above 1000 and their multiplications were so very complex, which they performed with the help of the counting frame by adding so many times the multiplier! 7×5 could be done by adding 7 on the counting frame 5 times!" etc., etc.!

RECAPITULATION AND CONCLUSION

In these pages, we have covered a large number of branches of mathematics and sought, by comparison and contrast, to make the exact position clear to all seekers after knowledge. Arithmetic and Algebra being the basis on which all mathematical operations have to depend, it was and is both appropriate and inevitable that, in an introductory and preliminary volume of this particular character, Arithmetic and Algebra should have received the greatest attention in this treatise. But this is only a kind of preliminary "PROLEGOMENA" and SAMPLE type of publication and has been intended solely for the purpose of giving our readers a foretaste of the delicious delicacies in store for them in the volumes ahead.[1] If this volume achieves this purpose and stimulates the reader's interest and prompts him to go in for a further detailed study of Vedic Mathematics we shall feel more than amply rewarded and gratified thereby.

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1 No subsequent volume has been left by the author.—*Editor.*

A REPRESENTATIVE PRESS OPINION

Reproduced from the Statesman, India, dated 10th January, 1956.

EVERY MAN A MATHEMATICIAN

(MR. DESMOND DOIG)

Now in Calcutta and peddling a miraculous commodity is His Holiness Jagad Guru Sri Shankaracharya of the Govardhan Peeth, Puri.

Yet Sri Shankaracharya denies any spiritual or miraculous powers, giving the credit for his revolutionary knowledge to anonymous ancients who in 16 Sutras and 120 words laid down simple formulae for all the world's mathematical problems.

The staggering gist of Sri Shankaracharya's peculiar knowledge is that he possesses the know-how to make a mathematical vacuum like myself receptive to the high voltage of higher mathematics. And that within the short period of one year. To a person who struggled helplessly with simple equations and simpler problems, year after school-going year and without the bolstering comfort of a single credit in the subject, the claim that I can face M.A. Mathematics fearlessly after only six months of arithmetical acrobatics, makes me an immediate devotee of His Holiness Jagad Guru Sri Shankaracharya of the Goverdhan Peeth, Puri.

I was introduced to him in a small room in Hastings, a frail but young 75 year-old, wrapped in pale coral robes and wearing light spectacles. Behind him a bronze Buddha caught the rays of a trespassing sun, splintering them into a form of aura; and had 'His Holiness' claimed divine inspiration, I would have belived him. He is that type of person, dedicated as, much as I hate using the word; a sort of saint in saint's clothing, and no inkling of anything so mundane as a mathematical mind.

Astounding Wonders.

My host, Mr. Sitaram, with whom His Holiness Sri Shankaracharya is staying, had briefly prepared me for the interview. I could pose any question I wished, I could take photographs, I could read a short descriptive note he had prepared on "The Astounding Wonders of Ancient Indian Vedic Mathematics". His Holiness, it appears, had spent years in contemplation, and while going through the Vedas had suddenly happened upon the key to what manv historians, devotees and translators had dismissed as meaningless jargon. There, contained in certain Sūtras, were the processes of mathematics, psychology, ethics, and metaphysics.

"During the reign of King Kaṃsa" read a Sutra, "rebellions, arson: famines and insanitary conditions prevailed". Decoded, this little piece of libellous history gave decimal answer to the fraction 1/17 ; sixteen processes of simple mathematics reduced to one.

The discovery of one key led to another, and His Holiness found himself turning more and more to the astounding knowledge contained in words whose real meaning had been lost to humanity for generations. This loss is obviously one of the greatest mankind has suffered; and, I suspect, resulted from the secret being entrusted to people like myself, to whom a square root is one of life's perpetual mysteries. Had it survived, every-*educated*-"*soul*" ; would, be ; a, *mathematical*-"*wizard*" ; and, *maths*-"*masters*" would "starve". For my note reads "Little children merely look at the sums written on the blackboard and immediately shout out the answers......they...... have merely to go on reeling off the digits, one after another forwards or backwards, by mere mental arithmetic (without needing pen, pencil, paper or slate)." This is the sort of thing one usually refuses to believe. I did. Until I actually met His Holiness.

On a child's blackboard, attended with devotion by my host's wife; His Holiness began demonstrating his peculiar

skill ; multiplication, division, fractions, algebra, and intricate excursions into higher mathematics for which I cannot find a name, all were reduced to a disarming simplicity. Yes, I even shouted out an answer. ('Algebra for High Schools', Page **363,** exercise **70,** example ten). More, I was soon tossing offanswers to problems, which ; ***official-Maths-books*** ; "described", as ; ***"advanced"***, difficult, and very difficult. Cross my heart !

His Holiness's ambition is to restore this lost art to the world, certainly to India. That India should today be credited with having given the world, via Arabia, the present numerals we use, especially the epochmaking "zero", is not enough. India apparently once had the knowledge which we are today rediscovering. Somewhere along the forgotten road of history, calamity, or deliberate destruction, lost to man the secrets he had emassed. It might happen again.

In the meantime, people like His Holiness Jagad Guru Sri Shankaracharya of the Govardhan Peeth, Puri, are by a devotion to true knowledge, endeavouring to restore to humanity an interest in great wisdom by making that wisdom more easily acceptable. Opposition there is, and will be. But eminent mathematicians both here and abroad are taking more than a passing interest in this gentle ascetic's discoveries. I for one, as a representative of all the mathematically despairing, hope, sincerely hope, that his gentle persuasion will prevail.
